Essays on Strategic Trading

by

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Abstract

Institutional investors, such as banks and hedge funds, typically base their trading strategies on stochastic signals extracted from data sets. This thesis explores the determinants of an optimal trading strategy in a game theoretical framework. Since institutional investors compete for trading profits, they have an intrinsic motive to forecast their competitors' information and incorporate that forecast into their trading strategies. This thesis contributes to the literature by letting trading strategies account for stylized empirical facts like autocorrelated order flows and news arrivals.

The first chapter explores the influence of autocorrelated order flows on trading strategies. Since market microstructure models predict that informed traders have an intrinsic motive to submit unpredictable orders, the empirically observed autocorrelation in order flows is likely to come from uninformed traders. Instead of assuming uninformed orders to be white noise, the model allows uninformed orders to exhibit a general correlation structure that generates autocorrelation in the aggregate order flow. As a result, the market maker's prices and beliefs are linear functions of the innovation in the aggregate order flow, implying that any form of predictability in financial markets gets priced immediately. A vital result of the model is that autocorrelation in the uninformed order flow implies uninformed orders to be conditionally correlated with the asset value and informed traders' signals. Therefore, autocorrelated uninformed orders also contain relevant information upon conditioning. Although informed traders still submit unpredictable orders in equilibrium, a numerical analysis shows that the correlation structure

of uninformed orders significantly affects informed traders' trading intensities and expected profits.

The second chapter explores the influence of news arrivals on trading strategies. Given that modern market makers are high-frequency traders competing for speed, the model assumes that the market maker can trade news faster than other traders. Consequently, the market maker is also an informed trader since she possesses short-lived private information about the asset value in every period. Informed traders learn the news after the market maker and incorporate that information into their trading strategies. A vital result of the model is that the market maker only prices the innovation in news. In general, news improves price discovery and has the highest price impact the first time it arrives. Also, since news arrivals reveal information about the asset value, informed traders' expected profits are negatively related to the informativeness of news. Finally, depending on the correlation structure of news arrivals, the model can produce new stylized facts like negative trading intensities and increasing price sensitivities to news.

Contents

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1 Information in Noise: Strategic Trading under Autocorrelated Uninformed Orders

Vladislav Gounas[†]

Abstract

This paper develops a strategic trading model in which uninformed orders exhibit a general correlation structure that generates autocorrelation in the aggregate order flow. Since the order flow is predictable, informed traders and the market maker not only need to infer information about the asset value but also forecast future order flows. The correlation structure of uninformed orders significantly affects trading intensities, market liquidity, and price efficiency. Since the empirical autocorrelation in order flows is likely to come from uninformed traders, strategic trading models should not assume them to be simply noise.

Keywords: Kyle model, autocorrelated order flow, strategic trading, asymmetric information, signal processing, price impact, market microstructure.

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1.1 Introduction

Strategic trading models usually classify traders as either informed or uninformed.[1] The common assumption is that uninformed traders are noise traders who submit orders completely randomly due to liquidity or rebalancing needs. However, this is a relatively strong assumption. Just because most traders do not possess non-public information does not imply that their trades should follow a random walk. So what if orders submitted by uninformed traders are not simply noise? What if instead there is a pattern to uninformed orders?

The first strategic trading model was developed by Kyle (1985), and ever since its original publication, the model has inspired a large body of literature.[2] A key result in strategic trading models with long-lived private information is that informed traders have an intrinsic motive to place unpredictable orders. Due to informed traders' trading strategy, the market cannot infer whether informed traders will submit buy or sell orders. Since orders by uninformed traders are usually assumed to be white noise, the aggregate order flow is uncorrelated over time in those models.

However, as studies by Hasbrouck (1991a,b) and Brogaard, Hendershott, and Riordan (2019) show, there is significant autocorrelation in empirical order flows. Consequently, if informed traders have an incentive to be unpredictable with their trades, then the empirical autocorrelation should come from uninformed traders. Consider momentum strategies as a simple example. If a single stock performed strongly in the past, a momentum strategy would imply buying that stock. Thus,

[1] The existence of uninformed traders is a necessary condition for informed traders to participate in the market. See Kyle (1985), Glosten and Milgrom (1985), and Easley and O'Hara (1987).

[2] Extensions with multiple informed traders include Holden and Subrahmanyam (1992, 1994), Foster and Viswanathan (1994, 1996), Bernhardt and Miao (2004), Dridi and Germain (2009), Colla and Mele (2010), Ostrovsky (2012), Rostek and Weretka (2012), Lambert, Ostrovsky, and Panov (2018), and Sastry and Thompson (2019). Back (1992), Back, Cao, and Willard (2000), Back and Baruch (2004), and Collin-Dufresne and Fos (2016) extend the Kyle (1985) model in continuous time. Admati and Pfleiderer (1988), Kyle (1989), Seppi (1990), Foster and Viswanathan (1990), Degryse, Jong, and Kervel (2014), and Choi, Larsen, and Seppi (2019) incorporate strategic uninformed traders into their models. Caballe and Krishnan (1994) and Pasquariello (2007) study a multi-asset version of the one-period Kyle (1985) model with multiple informed traders.

if several uninformed traders were to follow this strategy, not only would the order flow be autocorrelated, but it might also be correlated with the true asset value and even non-public signals.

Moreover, there are also economic reasons why the autocorrelation in the order flow is likely to come from uninformed traders. In Degryse, Jong, and Kervel (2014) and Choi, Larsen, and Seppi (2019), autocorrelation in the order flow arises endogenously by modeling an uninformed trader who strategically rebalances his position to maintain a target level.[3] Since that uninformed trader must reach his target level before the end of the trading day, his orders are autocorrelated as a result. To conclude, it stands to reason that uninformed traders generate autocorrelation in the order flow and that their orders might even be correlated with other variables of interest.

This paper contributes to the literature by allowing uninformed orders to exhibit autocorrelation in the time series and cross-sectional correlation with the asset value and informed traders' signals. As a result, the aggregate order flow will also be autocorrelated and thus predictable. Even though this paper speaks of uninformed traders, one can think of them as being partially informed traders who do not behave strategically. Being uninformed in that context means not possessing relevant non-public information. This paper wants to ask the following questions: how should heterogeneously informed traders adjust their trading strategies under (auto)correlated uninformed orders? How does the (auto)correlation affect the value of non-public information? What is the impact on information efficiency in financial markets? How sensitive are prices to arriving orders when there is autocorrelation in the order flow?

This paper builds on the model by Foster and Viswanathan (1996), who were the first to derive a dynamic trading equilibrium with heterogeneously informed traders.[4] As is usual in strategic trading models, it is assumed that the market

[3] Choi, Larsen, and Seppi (2019) also allow the uninformed trader to be partially informed. However, the model still generates autocorrelation in the order flow if the uninformed trader is entirely uninformed.

[4] Back, Cao, and Willard (2000) study the Foster and Viswanathan (1996) model in continuous

consists of informed traders, a mass of uninformed traders, and a competitive market maker. Before trading starts, each informed trader receives a private signal about the asset's true value, leading to a "forecasting the forecasts of others" problem. Foster and Viswanathan (1996) show that the average of the informed traders' signals is a sufficient statistic to forecast the asset value. However, that result no longer holds once uninformed orders are allowed to be (auto)correlated. Since uninformed orders now also contain information about the asset value, the average signal no longer captures all available information in the market.

A vital result of this paper's model is that autocorrelation in the uninformed order flow implies uninformed orders to be conditionally correlated with the asset value and informed traders' signals in the cross-section. Thus, even if autocorrelated uninformed orders do not initially contain information about the asset value and non-public signals, they do so upon conditioning. Another main result is that only the innovation in the aggregate order flow has a price impact, implying that any form of predictability in the market gets priced immediately.

A numerical study shows that (auto)correlated uninformed orders significantly affect informed traders' expected profits and price discovery. Informed traders make the highest expected profits, and prices are least revealing if uninformed orders are negatively correlated with the asset value in the cross-section. In that scenario, private information is most valuable because uninformed traders effectively conceal informed traders' orders by pushing the price away from the true asset value. In contrast, informed traders' expected profits are lowest, and price efficiency is highest if the cross-sectional correlation between uninformed orders and the asset value is positive. In that case, uninformed orders reveal information about the asset's true value, generating competition for informed traders.

Moreover, the speed at which information gets incorporated into prices depends on the cross-sectional correlation between uninformed orders and informed traders'

time. Other papers with heterogeneously informed traders include Foster and Viswanathan (1994), Bernhardt and Miao (2004), Dridi and Germain (2009), Colla and Mele (2010), Rostek and Weretka (2012), Ostrovsky (2012), and Lambert, Ostrovsky, and Panov (2018).

signals. A positive correlation implies that most information about the asset value gets revealed in the first periods. In contrast, price discovery is initially low and increases in later periods if the correlation is negative.

Finally, the shape of the market maker's illiquidity parameter is determined by the correlation structure of uninformed orders. In particular, the market maker's price sensitivity to order flows can be monotonically decreasing/increasing, constant, U-shaped, and even have an inflection point. Thus, (auto)correlated uninformed orders can explain various dynamics of market depth.

This research is related to Aase, Bjuland, and Øksendal (2012) and Lambert, Ostrovsky, and Panov (2018). Aase, Bjuland, and Øksendal (2012) consider a model with one perfectly informed trader and allow the uninformed order flow to be correlated with the asset value. The authors find a negative relationship between the informed trader's expected profit and the correlation between uninformed orders and the asset value. Additionally, price efficiency is lowest if uninformed orders are negatively correlated with the asset value.

In contrast, Lambert, Ostrovsky, and Panov (2018) consider multiple informed traders with heterogeneous information and allow for a general correlation structure between the asset value, informed traders' signals, and the uninformed order flow.[5] They also find that correlated uninformed orders significantly affect informed traders' expected profits and price discovery. Moreover, the authors provide general results on equilibrium existence and uniqueness and derive general closed-form equilibrium characterizations.

However, both papers focus on a one-period model, whereas this paper wants to study a dynamic trading game. Even though it will be necessary to impose symmetry conditions for (co)variances to keep the analysis tractable in the dynamic setting, this paper can be viewed as a multi-period extension of the one-period model in Lambert, Ostrovsky, and Panov (2018).

The rest of this paper is organized as follows. Section 1.2 explains the gen-

[5] Lambert, Ostrovsky, and Panov (2018) also allow the market maker to receive a signal that is correlated with the asset value, informed traders' signals, and the uninformed order flow.

eral model framework, and Section 1.3 describes the conjectured equilibrium. In Section 1.4, informed traders' updating processes along equilibrium and off-equilibrium paths are analyzed. Moreover, necessary and sufficient conditions for a dynamic trading equilibrium are provided. Finally, Section 1.5 numerically evaluates the model, and Section 1.6 concludes.

1.2 The Model

Let the trading day be standardized to the interval $[0, 1]$. Trading takes place over $N \in \mathbb{N}$ sequential batch auctions at equidistant time points $1/N$. The market consists of $M \in \mathbb{N}$ risk-neutral informed traders, a mass of uninformed traders, and one risk-neutral competitive market maker, who all trade a single risky asset with stochastic value v and variance $\Sigma_0^v \equiv \text{Var}[v]$. Trading takes place as follows:

(i) In each period $n \in \{1, \ldots, N\}$, informed and uninformed traders place their orders to the market maker.

(ii) The market maker observes the aggregate order flow and sets the clearing price.

(iii) The clearing price and aggregate order flow become public information.

(iv) Steps (i) to (iii) are repeated until the last trading period N is reached.

(v) After the last round of trading, the liquidation value of the asset v gets publicly announced, and profits of the asset holders are realized.

Before the start of the trading game, each informed trader $i \in \{1, \ldots, M\}$ receives a private signal $s_{i,0}$ about the asset's true value. Denote the signal vector as $\boldsymbol{s} \equiv [s_{1,0}, \ldots, s_{M,0}]'$, and let $\Sigma_0^{\boldsymbol{s}}$ be the corresponding covariance matrix with diagonals $\Sigma_0^{s_{i,0}} \equiv \text{Var}[s_{i,0}]$ and off-diagonals $\Sigma_0^{s_{i,0},s_{j,0}} \equiv \text{Cov}[s_{i,0}, s_{j,0}]$ where $j \in \{1, \ldots, M\}$ and $j \neq i$. Also, define the covariance between the asset value and informed trader i's signal as $\Sigma_0^{v,s_{i,0}} \equiv \text{Cov}[v, s_{i,0}]$. To keep future analysis tractable, the following symmetry conditions are imposed for all $i, j, k \in \{1, \ldots, M\}$:

$$\Sigma_0^{s_{i,0}} = \Sigma_0^{s_{j,0}}, \quad \Sigma_0^{s_{i,0},s_{j,0}} = \Sigma_0^{s_{i,0},s_{k,0}}, \quad \Sigma_0^{s_{i,0},s_{j,0}} = \Sigma_0^{s_{k,0},s_{j,0}}, \quad \Sigma_0^{v,s_{i,0}} = \Sigma_0^{v,s_{j,0}}.$$

The symmetry in (co)variances is a crucial assumption for the analysis that follows as it will keep the dimensionality space small.

In each period $n \in \{1, \ldots, N\}$, uninformed traders submit the aggregate order u_n. Collect these orders into the vector $\boldsymbol{u} \equiv [u_1, \ldots, u_N]'$, and let $\Sigma_0^{\boldsymbol{u}}$ be the corresponding covariance matrix. Instead of assuming uninformed orders to be simply noise, they are allowed to be autocorrelated in the time series and exhibit cross-sectional correlation with the asset value and informed traders' signals.

To this end, assume that $\Sigma_0^{\boldsymbol{u}}$ has diagonals $\Sigma_0^{u_n} \equiv \mathrm{Var}[u_n]$ and off-diagonals $\Sigma_0^{u_n,u_m} \equiv \mathrm{Cov}[u_n, u_m]$ where $m \in \{1, \ldots, N\}$ and $m \neq n$. Moreover, define the covariance between the asset value and the uninformed order flow in period n as $\Sigma_0^{v,u_n} \equiv \mathrm{Cov}[v, u_n]$ and the covariance between informed trader i's signal and the uninformed order flow in period n as $\Sigma_0^{s_{i,0},u_n} \equiv \mathrm{Cov}[s_{i,0}, u_n]$. As before, to keep future analysis tractable, the following symmetry conditions are imposed for all $n, m, l \in \{1, \ldots, N\}$ and all $i, j \in \{1, \ldots, M\}$:

$$\Sigma_0^{u_n} = \Sigma_0^{u_m}, \quad \Sigma_0^{u_n,u_m} = \Sigma_0^{u_n,u_l}, \quad \Sigma_0^{u_n,u_m} = \Sigma_0^{u_l,u_m}, \quad \Sigma_0^{v,u_n} = \Sigma_0^{v,u_m}, \quad \Sigma_0^{s_{i,0},u_n} = \Sigma_0^{s_{j,0},u_m}.$$

Allowing for (auto)correlation in the uninformed order flow is the main contribution of this paper. At first glance, the uninformed order vector $\boldsymbol{u}$ could be interpreted as a signal vector just like $\boldsymbol{s}$. However, the crucial difference is that signals are realized simultaneously before the first trading period, whereas uninformed orders are only realized sequentially over time. Consequently, informed traders not only need to infer information about other informed traders' signals but also forecast future uninformed order flows.

Let $\Sigma_0^{v,\boldsymbol{s}}$ be the $(1 \times M)$ vector with entries $\Sigma_0^{v,s_{i,0}}$, $\Sigma_0^{v,\boldsymbol{u}}$ be the $(1 \times N)$ vector with entries Σ_0^{v,u_n}, and $\Sigma_0^{\boldsymbol{s},\boldsymbol{u}}$ be the $(M \times N)$ matrix with entries $\Sigma_0^{s_{i,0},u_n}$. It is assumed that the asset value, informed traders' signals, and uninformed orders are jointly normally distributed as follows:

$$\begin{pmatrix} v \\ \boldsymbol{s} \\ \boldsymbol{u} \end{pmatrix} \sim \mathcal{N}\left[\begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} \underset{(1\times 1)}{\Sigma_0^{v}} & \underset{(1\times M)}{\Sigma_0^{v,\boldsymbol{s}}} & \underset{(1\times N)}{\Sigma_0^{v,\boldsymbol{u}}} \\ \underset{(M\times 1)}{\Sigma_0^{\boldsymbol{s},v}} & \underset{(M\times M)}{\Sigma_0^{\boldsymbol{s}}} & \underset{(M\times N)}{\Sigma_0^{\boldsymbol{s},\boldsymbol{u}}} \\ \underset{(N\times 1)}{\Sigma_0^{\boldsymbol{u},v}} & \underset{(N\times M)}{\Sigma_0^{\boldsymbol{u},\boldsymbol{s}}} & \underset{(N\times N)}{\Sigma_0^{\boldsymbol{u}}} \end{pmatrix}\right]. \tag{1.1}$$

Denote the covariance matrix in (1.1) as $\boldsymbol{\Sigma_0}$ and assume the distribution is common knowledge in the market.[6]

Finally, there is a risk-neutral competitive market maker who observes the aggregate order flow and sets the clearing price. Suppose each informed trader $i \in \{1, \ldots, M\}$ submits a quantity $x_{i,n}$ in period $n \in \{1, \ldots, N\}$, then the aggregate order flow y_n in period n is defined as

$$y_n \equiv \sum_{i=1}^{M} x_{i,n} + u_n. \tag{1.2}$$

The market maker cannot distinguish whether a single order is informed or uninformed, and perfect competition implies that the clearing price p_n in period n is set to her conditional expectation of the asset value:

$$p_n \equiv \mathbb{E}\left[v \middle| y_{1:n}\right], \tag{1.3}$$

where the order flow history $y_{1:n} \equiv (y_1, \ldots, y_n)$ is the available information to the market maker after n periods of trading.

Consider the objective function of any informed trader $i \in \{1, \ldots, M\}$. In each period $n \in \{1, \ldots, N\}$, he maximizes his terminal conditional expected profit.

[6] Naturally, restrictions are required to ensure that the covariance matrix $\boldsymbol{\Sigma_0}$ in (1.1) is positive definite. Numerically, this will put upper bounds on the absolute values of covariances.

The Bellman equation for his dynamic programming problem writes

$$\max_{x_{i,n}} \mathbb{E}\left[(v - p_n)x_{i,n} + V_{i,n}\Big| s_{i,0}, y_{1:n-1}, \tilde{x}_{i,1:n-1}\right], \tag{1.4}$$

where

$$V_{i,n} \equiv \max_{x_{i,n+1},\dots,x_{i,N}} \mathbb{E}\left[\sum_{m=n+1}^{N}(v - p_m)x_{i,m}\Bigg| s_{i,0}, y_{1:n}, \tilde{x}_{i,1:n}\right] \tag{1.5}$$

is informed trader i's value function after n rounds of trading. Note that informed trader i's information set in period n consists of his private signal $s_{i,0}$, the order flow history $y_{1:n-1}$, and his individual trading history $\tilde{x}_{i,1:n-1}$. The individual trading history is written with a "tilde" to stress that the history can be arbitrary. By definition, the Bellman equation accounts for both optimal and suboptimal play in the past.

A Bayesian Nash equilibrium of the dynamic trading game is defined as in Foster and Viswanathan (1996).

Definition 1.1. *The strategies $\{x_{1,n}, \dots, x_{M,n}, p_n\}_{n=1}^N$ form a Bayesian Nash equilibrium of the dynamic trading game if they satisfy two conditions:*

(i) Given the strategies $\{x_{1,n}, \dots, x_{i-1,n}, x_{i+1,n}, \dots, x_{M,n}\}_{n=1}^N$ and $\{p_n\}_{n=1}^N$, the strategy $\{x_{i,n}\}_{n=1}^N$ maximizes the objective function (1.4) for each informed trader $i \in \{1, \dots, M\}$ in every period $n \in \{1, \dots, N\}$.

(ii) Given the strategies $\{x_{1,n}, \dots, x_{M,n}\}_{n=1}^N$, the market maker's pricing rule $\{p_n\}_{n=1}^N$ satisfies (1.3) in every period $n \in \{1, \dots, N\}$.

There are two important things to note about Definition 1.1. First, the Bellman equation (1.4) requires informed trader i's strategy $x_{i,n}$ in period n to be optimal given any arbitrary past trading history $\tilde{x}_{i,1:n-1}$. It is therefore necessary to distinguish between equilibrium and off-equilibrium play. Second, deviations by any informed trader $i \in \{1, \dots, M\}$ from his optimal strategy are unobservable to the market maker and other informed traders. Consequently, there are no off-equilibrium beliefs in the model.

1.3 The Conjectured Equilibrium

Following Kyle (1985) and its extensions, this paper wants to derive a linear and symmetric trading equilibrium. To this end, conjecture that informed traders' equilibrium strategies are of the same form as in Foster and Viswanathan (1996), that is, for all $i \in \{1, \ldots, M\}$ and all $n \in \{1, \ldots, N\}$:

$$x_{i,n} = \beta_n s_{i,n-1}, \tag{1.6}$$

where $s_{i,n-1} \equiv s_{i,0} - \mathbb{E}\left[s_{i,0}\middle|y_{1:n-1}\right]$ is the part of informed trader i's private information that is not known to the market maker after $n-1$ periods of trading. Thus, conjecture that even when uninformed orders are (auto)correlated, informed traders have an incentive to place orders that are unpredictable from the market maker's point of view. However, it is important to stress that (1.6) only characterizes the strategy played along the equilibrium path and does not yet account for off-equilibrium play.

Although informed traders' orders are unpredictable, the aggregate order flow is autocorrelated since

$$\mathbb{E}\left[y_n\middle|y_{1:n-1}\right] = \mathbb{E}\left[\sum_{i=1}^{M} \beta_n s_{i,n-1} + u_n\middle|y_{1:n-1}\right] = \mathbb{E}\left[u_n\middle|y_{1:n-1}\right], \tag{1.7}$$

that is, the autocorrelation in the aggregate order flow comes solely from uninformed traders. In particular, the trading strategy (1.6) has the following implications for the market maker's updating processes along equilibrium play.[7]

Lemma 1.1. *Suppose informed trading strategies are as in (1.6). Then for all $n \in \{1, \ldots, N\}$, $i \in \{1, \ldots, M\}$, and $t \in \{1, \ldots, N-n\}$, the market maker recursively updates her prices and beliefs as follows:*

$$p_n \equiv \mathbb{E}\left[v\middle|y_{1:n}\right] = p_{n-1} + \lambda_n(y_n - r_{n,n-1}), \tag{1.8}$$

[7] The market maker has no off-equilibrium beliefs since off-equilibrium play is unobservable.

$$t_{i,n} \equiv \mathbb{E}\left[s_{i,0} \middle| y_{1:n}\right] = t_{i,n-1} + \zeta_n(y_n - r_{n,n-1}), \tag{1.9}$$

$$r_{n+t,n} \equiv \mathbb{E}\left[u_{n+t} \middle| y_{1:n}\right] = r_{n,n-1} + \theta_n(y_n - r_{n,n-1}), \tag{1.10}$$

where λ_n, ζ_n, and θ_n are projection coefficients.

Proof. See Appendix 1.A.1.

There are two important things to note about Lemma 1.1. First, prices and beliefs are linear functions of $y_n - r_{n,n-1}$, the innovation in the aggregate order flow. Thus, the predictable part of the aggregate order flow $r_{n,n-1}$ has no price impact. Moreover, similar to Kyle (1985), the projection coefficient λ_n is an inverse measure of market depth. The less sensitive prices are to aggregate order flow innovations, the deeper the market.

Second, the market maker needs to forecast future uninformed orders. According to (1.7), this problem is equivalent to forecasting future aggregate order flows. It can be inferred from (1.10) that $r_{n+t,n}$ is independent of the forecasting horizon t. Consequently, forecasting uninformed orders only one period ahead is sufficient, that is, after n periods, $r_{n+1,n}$ is a sufficient statistic for forecasting all future uninformed orders. This property is important since it keeps the dimensionality space small. Otherwise, the market maker would need to forecast each future uninformed order flow separately, resulting in a dimensionality problem with a growing number of trading periods.

While the market maker can also smooth past and filter current uninformed orders, it is not necessary for solving the model. More importantly, the smoothing and filtering processes are different for each uninformed order flow, that is, for all $n \in \{1, \ldots, N\}$, one can show that $r_{m,n} \neq r_{l,n}$ for all $m, l \in \{1, \ldots, n\}$ where $m \neq l$. Thus, to avoid the curse of dimensionality, this paper does not keep track of the smoothing and filtering processes.[8]

[8] One can show that $\mathbb{E}\left[v \middle| \boldsymbol{s}, \boldsymbol{u}\right] = \mu \sum_{i=1}^{M} s_{i,0} + \omega \sum_{n=1}^{N} u_n \equiv \hat{v}$, where μ and ω are constants. While $\hat{v}$ is a sufficient statistic for solving the model, it also leads to a dimensionality issue since one would need to smooth past and filter current uninformed orders.

It will prove helpful to consider uncertainty from the market maker's point of view. To this end, define for all periods $n \in \{1, \dots, N\}$ and all informed traders $i, j \in \{1, \dots, M\}$ the following conditional (co)variances:

$$
\begin{aligned}
\Sigma_n^{v} &\equiv \mathrm{Var}\left[v \middle| y_{1:n}\right], & \Sigma_n^{s_{i,0}} &\equiv \mathrm{Var}\left[s_{i,0} \middle| y_{1:n}\right], \\
\Sigma_n^{u_{n+1}} &\equiv \mathrm{Var}\left[u_{n+1} \middle| y_{1:n}\right], & \Sigma_n^{v,s_{i,0}} &\equiv \mathrm{Cov}\left[v, s_{i,0} \middle| y_{1:n}\right], \\
\Sigma_n^{v,u_{n+1}} &\equiv \mathrm{Cov}\left[v, u_{n+1} \middle| y_{1:n}\right], & \Sigma_n^{s_{i,0},s_{j,0}} &\equiv \mathrm{Cov}\left[s_{i,0}, s_{j,0} \middle| y_{1:n}\right], \\
\Sigma_n^{s_{i,0},u_{n+1}} &\equiv \mathrm{Cov}\left[s_{i,0}, u_{n+1} \middle| y_{1:n}\right], & \Sigma_n^{u_{n+1},u_{n+2}} &\equiv \mathrm{Cov}\left[u_{n+1}, u_{n+2} \middle| y_{1:n}\right].
\end{aligned}
$$

These (co)variances are conditional on the market maker's information after n periods, consisting of the aggregate order flow history $y_{1:n}$. In particular, Σ_n^v measures how much uncertainty about the asset value remains after n trading rounds. The following lemma shows how these conditional (co)variances are related.

Lemma 1.2. *Suppose informed trading strategies are as in (1.6). Then for all $n \in \{1, \dots, N-1\}$ and $i, j \in \{1, \dots, M\}$, it holds that*

$$\Sigma_{n-1}^{s_{i,0}} - \Sigma_n^{s_{i,0}} = \Sigma_{n-1}^{s_{i,0},s_{j,0}} - \Sigma_n^{s_{i,0},s_{j,0}}, \tag{1.11}$$

$$\Sigma_{n-1}^{u_n} - \Sigma_n^{u_{n+1}} = \Sigma_{n-1}^{u_n,u_{n+1}} - \Sigma_n^{u_{n+1},u_{n+2}}. \tag{1.12}$$

Moreover, the market maker's illiquidity parameter λ_n in (1.8) satisfies

$$\lambda_n = \begin{pmatrix} \Sigma_{n-1}^{v,\boldsymbol{s}} & \Sigma_{n-1}^{v,u_n} \end{pmatrix} \begin{pmatrix} \Sigma_{n-1}^{\boldsymbol{s}} & \Sigma_{n-1}^{\boldsymbol{s},u_n} \\ \Sigma_{n-1}^{u_n,\boldsymbol{s}} & \Sigma_{n-1}^{u_n} \end{pmatrix}^{-1} \begin{pmatrix} \zeta_n \\ \vdots \\ \zeta_n \\ \tilde{\theta}_n \end{pmatrix}, \tag{1.13}$$

where $\Sigma_{n-1}^{\boldsymbol{s}}$ is the $(M \times M)$ conditional covariance matrix with diagonals $\Sigma_{n-1}^{s_{i,0}}$ and off-diagonals $\Sigma_{n-1}^{s_{i,0},s_{j,0}}$, $\Sigma_{n-1}^{v,\boldsymbol{s}}$ is the $(1 \times M)$ conditional covariance vector with entries $\Sigma_{n-1}^{v,s_{i,0}}$, $\Sigma_{n-1}^{\boldsymbol{s},u_n}$ is the $(M \times 1)$ conditional covariance vector with entries $\Sigma_{n-1}^{s_{i,0},u_n}$, and $\tilde{\theta}_n \neq \theta_n$ is a projection coefficient.

Proof. See Appendix 1.A.2.

The first part of Lemma 1.2 implies for all periods $n \in \{1, \ldots, N-1\}$:

$$\Sigma_{n-1}^{s_{i,0}} - \Sigma_{n-1}^{s_{i,0},s_{j,0}} \equiv \chi_s, \tag{1.14}$$

$$\Sigma_{n-1}^{u_n} - \Sigma_{n-1}^{u_n,u_{n+1}} \equiv \chi_u, \tag{1.15}$$

where χ_s and χ_u are constants. Thus, for both informed traders' signals and future uninformed orders, the difference between variances and covariances is independent of the trading period.

The second part of Lemma 1.2 implies that the market maker's update of the asset value is not proportional to her update of informed traders' signals:

$$p_n - p_{n-1} \not\propto \sum_{i=1}^{M} (t_{i,n} - t_{i,n-1}). \tag{1.16}$$

A key result in Foster and Viswanathan (1996) is that the market maker only needs to forecast the average of the informed traders' signals. However, that result is no longer true once (auto)correlation in the uninformed order flow is introduced. Since uninformed orders now also contain relevant information, the average signal is no longer a sufficient statistic for all available information in the market. Thus, (auto)correlated uninformed orders disentangle the one-to-one relationship between forecasting the asset value and forecasting the private information of informed traders.

1.4 Informed Traders' Updating Processes

This section addresses the dimensionality issue of the dynamic trading game and shows how informed traders' beliefs are updated over time. Definition 1.1 requires one to distinguish between updating along equilibrium and off-equilibrium play. Deriving the updating processes will help identify the state variables of informed traders' dynamic programming problem.

1.4.1 Updating along the Equilibrium Path

Following Foster and Viswanathan (1996), it is first shown that for every informed trader $i \in \{1, \ldots, M\}$ the individual past trading history is redundant information along the equilibrium path. Indeed, from (1.6), (1.9), and (1.10), one can infer that informed trader i's optimal trading strategy $x_{i,n}$ in every period $n \in \{1, \ldots, N\}$ is equivalent to

$$
\begin{aligned}
x_{i,n} = \beta_n s_{i,n-1} = \beta_n(s_{i,0} - t_{i,n-1}) &= \beta_n \left(s_{i,0} - \sum_{t=1}^{n-1} \zeta_t \Big[y_t - r_{t,t-1} \Big] \right) \\
&= \beta_n \left(s_{i,0} - \sum_{t=1}^{n-1} \zeta_t \left[y_t - \sum_{r=1}^{t-1} \left(\prod_{s=r+1}^{t-1} \Big[1 - \theta_s \Big] \right) \theta_r y_r \right] \right).
\end{aligned} \tag{1.17}
$$

Equation (1.17) implies that informed trader i's optimal trading strategy in period n is a function of his private signal $s_{i,0}$ and the past aggregate order flow history $y_{1:n-1}$. Therefore, the individual trading history $x_{i,1:n-1}$ is redundant and the meaningful information for informed trader i is just $(s_{i,0}, y_{1:n-1})$. This result also implies that the only relevant information that informed trader i has relative to the market maker is his private signal $s_{i,0}$, which helps identify the state variables for updating along the equilibrium path.

Lemma 1.3. *Assume the market maker sets prices and forms beliefs as in (1.8), (1.9), and (1.10), and all informed traders $i, j \in \{1, \ldots, M\}$ where $i \neq j$ trade as in (1.6). Then $s_{i,n-1}$ is a sufficient statistic for informed trader i's updating processes after $n-1$ periods for all $n \in \{1, \ldots, N\}$. In particular, the updating processes satisfy*

$$
\mathbb{E}\left[v - p_{n-1} \middle| s_{i,0}, y_{1:n-1}, x_{i,1:n-1} \right] = \eta_n s_{i,n-1}, \tag{1.18}
$$

$$
\mathbb{E}\left[s_{j,n-1} \middle| s_{i,0}, y_{1:n-1}, x_{i,1:n-1} \right] = \phi_n s_{i,n-1}, \tag{1.19}
$$

$$
\mathbb{E}\left[u_n - r_{n,n-1} \middle| s_{i,0}, y_{1:n-1}, x_{i,1:n-1} \right] = \psi_n s_{i,n-1}, \tag{1.20}
$$

where η_n, ϕ_n, and ψ_n are projection coefficients.

Proof. See Appendix 1.A.3.

If all informed traders (including i) have played optimally in the first $n-1$ trading rounds, then the residual signal $s_{i,n-1} = s_{i,0} - t_{i,n-1}$ is a sufficient statistic for informed trader i to forecast the asset value, other informed traders' signals, and the next period's uninformed order flow.[9] Thus, $s_{i,n-1}$ is the only state variable along equilibrium play. However, it is important to stress that these results only hold if informed trader i has played his equilibrium strategy in the first $n-1$ trading periods. Additional state variables are required to account for off-equilibrium play.

1.4.2 Updating along Off-Equilibrium Paths

Without loss of generality, assume informed trader i has submitted an arbitrary order sequence $\tilde{x}_{i,1:n}$ in the first $n \in \{1, \ldots, N\}$ trading periods while all other informed traders $j \in \{1, \ldots, M\}$ where $j \neq i$ follow the equilibrium strategy (1.6). Furthermore, suppose the market maker sets prices and forms beliefs as in (1.8), (1.9), and (1.10).

Since deviations from the equilibrium strategy are unobservable, informed trader i can manipulate the market maker's and other informed traders' beliefs about the asset value, signals, and uninformed orders. More importantly, if informed trader i manipulates beliefs by submitting arbitrary orders $\tilde{x}_{i,1:n}$, the projection theorem for Gaussian random variables cannot be applied since $s_{i,n}$ and $y_{1:n}$ need not be jointly normally distributed. In particular, $s_{i,n}$ will no longer be a sufficient forecasting statistic for informed trader i after n periods of trading.

To address this issue, closely follow Foster and Viswanathan (1996) and introduce the following notation for all trading periods $n \in \{1, \ldots, N\}$ and informed traders $j \in \{1, \ldots, M\}$:

$$\hat{y}^i_n \equiv \sum_{j=1}^{M} \beta_n \hat{s}^i_{j,n-1} + u_n,$$

$$\hat{s}^i_{j,n} \equiv s_{j,0} - \hat{t}^i_{j,n} \text{ and } \hat{s}^i_{j,0} \equiv s_{j,0},$$

[9]Technically, informed traders need to form beliefs about all future uninformed order flows. However, as was shown in Lemma 1.1, it is sufficient to forecast uninformed orders only one period ahead.

$$\hat{p}^i_n \equiv \hat{p}^i_{n-1} + \lambda_n \left(\hat{y}^i_n - \hat{r}^i_{n,n-1}\right) \text{ and } \hat{p}^i_0 \equiv 0,$$

$$\hat{t}^i_{j,n} \equiv \hat{t}^i_{j,n-1} + \zeta_n \left(\hat{y}^i_n - \hat{r}^i_{n,n-1}\right) \text{ and } \hat{t}^i_{j,0} \equiv 0,$$

$$\hat{r}^i_{n+1,n} \equiv \hat{r}^i_{n,n-1} + \theta_n \left(\hat{y}^i_n - \hat{r}^i_{n,n-1}\right) \text{ and } \hat{r}^i_{1,0} \equiv 0.$$

These "hat" processes are the quantities that would have occurred along the equilibrium path in trading period n if informed trader i had followed his equilibrium strategy $\{\beta_m \hat{s}^i_{i,m-1}\}^n_{m=1}$. All results from the previous sections apply to these "hat" processes, which denote the equilibrium path processes. With this notation in mind, an important result from Foster and Viswanathan (1996), which will help characterize off-equilibrium paths, is stated.

Lemma 1.4. *Assume the market maker sets prices and forms beliefs as in (1.8), (1.9), and (1.10), and all informed traders $j \in \{1, \ldots, M\}$ where $j \neq i$ trade as in (1.6). Suppose informed trader i has submitted an arbitrary sequence of orders $\tilde{x}_{i,1:n}$ in the first $n \in \{1, \ldots, N\}$ trading periods. Then the following two information sets are equivalent:*

$$\left(s_{i,0}, y_{1:n}, \tilde{x}_{i,1:n}\right) \equiv \left(s_{i,0}, \hat{y}^i_{1:n}, \tilde{x}_{i,1:n}\right), \tag{1.21}$$

where $\hat{y}^i_{1:n} \equiv (\hat{y}^i_1, \ldots, \hat{y}^i_n)$.

Proof. See Appendix 1.A.4.

Lemma 1.4 states that the equilibrium order flow history $\hat{y}^i_{1:n}$ is in informed trader i's information set even along off-equilibrium paths. Intuitively, informed trader i knows how his trades affect the market maker's and other informed traders' strategies and beliefs. Consequently, he can recursively reconstruct the aggregate order flow history that would have occurred along the equilibrium path if he had played his equilibrium strategy. Since $\hat{y}^i_{1:n}$ is in informed trader i's information set, so is the history of the equilibrium quantities $\{\hat{s}^i_{i,m}, \hat{p}^i_m, \hat{t}^i_{j,m}, \hat{r}^i_{m+1,m}\}^n_{m=0}$.

Lemma 1.3 has shown that, along the equilibrium path, $\hat{s}^i_{i,n-1}$ is a sufficient forecasting statistic for informed trader i after $n-1$ periods of trading. The following lemma lists the additional state variables required for updating along off-equilibrium paths.

Lemma 1.5. *Assume the market maker sets prices and forms beliefs as in (1.8), (1.9), and (1.10), and all informed traders $j \in \{1, \ldots, M\}$ where $j \neq i$ trade as in (1.6). Suppose informed trader i has submitted arbitrary orders $\tilde{x}_{i,1:n-1}$ in the first $n-1$ trading rounds. Then $\hat{s}^i_{i,n-1}$, $\hat{p}^i_{n-1} - p_{n-1}$, $\hat{t}^i_{j,n-1} - t_{j,n-1}$, and $\hat{r}^i_{n,n-1} - r_{n,n-1}$ are sufficient statistics for informed trader i's updating processes after $n-1$ periods for all $n \in \{2, \ldots, N\}$. In particular, the updating processes satisfy*

$$\mathbb{E}\left[v - p_{n-1} \middle| s_{i,0}, y_{1:n-1}, \tilde{x}_{i,1:n-1}\right] = \eta_n \hat{s}^i_{i,n-1} + \left(\hat{p}^i_{n-1} - p_{n-1}\right), \tag{1.22}$$

$$\mathbb{E}\left[s_{j,n-1} \middle| s_{i,0}, y_{1:n-1}, \tilde{x}_{i,1:n-1}\right] = \phi_n \hat{s}^i_{i,n-1} + \left(\hat{t}^i_{j,n-1} - t_{j,n-1}\right), \tag{1.23}$$

$$\mathbb{E}\left[u_n - r_{n,n-1} \middle| s_{i,0}, y_{1:n-1}, \tilde{x}_{i,1:n-1}\right] = \psi_n \hat{s}^i_{i,n-1} + \left(\hat{r}^i_{n,n-1} - r_{n,n-1}\right). \tag{1.24}$$

Proof. See Appendix 1.A.5.

In Foster and Viswanathan (1996), the residual signal $\hat{s}^i_{i,n-1}$ and the price deviation from equilibrium $\hat{p}^i_{n-1} - p_{n-1}$ are sufficient statistics for updating along off-equilibrium paths. However, in the presence of (auto)correlated uninformed orders, two additional state variables are required: the deviation in beliefs from equilibrium about informed traders' signals $\hat{t}^i_{j,n-1} - t_{j,n-1}$ and the deviation in beliefs from equilibrium about the next period's uninformed order flow $\hat{r}^i_{n,n-1} - r_{n,n-1}$.

If uninformed orders are assumed to be white noise, the updating process of the asset value is proportional to the updating process of informed traders' signals. For this reason, $\hat{t}^i_{j,n-1} - t_{j,n-1}$ is a redundant state variable in Foster and Viswanathan (1996). However, as shown in (1.16), this proportionality breaks down if uninformed orders exhibit a general correlation structure. Consequently, updates about the asset value and updates about informed traders' signals have to be distinguished from each other.

1.4.3 Necessary and Sufficient Conditions for Equilibrium

With the insights from Lemma 1.5 in mind, conjecture informed trader i's optimal strategy, which accounts for both equilibrium and off-equilibrium play, to be linear in the state variables. In particular, given any arbitrary order sequence $\tilde{x}_{i,1:n-1}$, conjecture informed trader i's optimal strategy for the remaining trading periods $m \in \{n, \ldots, N\}$ to be of the following form:

$$x_{i,m} = \beta_m \hat{s}^i_{i,m-1} + \gamma_m \left(\hat{p}^i_{m-1} - p_{m-1}\right) + \alpha_m \left(\hat{t}^i_{j,m-1} - t_{j,m-1}\right) + \delta_m \left(\hat{r}^i_{m,m-1} - r_{m,m-1}\right). \tag{1.25}$$

Along the equilibrium path, one has $p_{m-1} = \hat{p}^i_{m-1}$, $t_{j,m-1} = \hat{t}^i_{j,m-1}$, and $r_{m,m-1} = \hat{r}^i_{m,m-1}$ so that only $\beta_m \hat{s}^i_{i,m-1}$ remains. Thus, the initially conjectured strategy (1.6), which will be played along the equilibrium path, is consistent with the general strategy (1.25).

Moreover, since informed traders are assumed to be risk-neutral, conjecture the value function $V_{i,n-1}$ of informed trader i after $n-1$ trading periods to be quadratic in the state variables, that is, for all periods $n \in \{1, \ldots, N\}$ and all informed traders $i, j \in \{1, \ldots, M\}$ where $i \neq j$, conjecture:

$$\begin{aligned} V_{i,n-1} = {} & a_{n-1} \left(\hat{s}^i_{i,n-1}\right)^2 + b_{n-1} \left(\hat{p}^i_{n-1} - p_{n-1}\right)^2 + c_{n-1} \left(\hat{t}^i_{j,n-1} - t_{j,n-1}\right)^2 \\ & + d_{n-1} \left(\hat{r}^i_{n,n-1} - r_{n,n-1}\right)^2 + e_{n-1} \hat{s}^i_{i,n-1} \left(\hat{p}^i_{n-1} - p_{n-1}\right) \\ & + f_{n-1} \hat{s}^i_{i,n-1} \left(\hat{t}^i_{j,n-1} - t_{j,n-1}\right) + g_{n-1} \hat{s}^i_{i,n-1} \left(\hat{r}^i_{n,n-1} - r_{n,n-1}\right) \\ & + h_{n-1} \left(\hat{p}^i_{n-1} - p_n\right) \left(\hat{t}^i_{j,n-1} - t_{j,n-1}\right) + i_{n-1} \left(\hat{p}^i_{n-1} - p_{n-1}\right) \left(\hat{r}^i_{n,n-1} - r_{n,n-1}\right) \\ & + j_{n-1} \left(\hat{t}^i_{j,n-1} - t_{j,n-1}\right) \left(\hat{r}^i_{n,n-1} - r_{n,n-1}\right) + k_{n-1}. \end{aligned} \tag{1.26}$$

Only the first and last terms of the value function remain along the equilibrium path. However, since Definition 1.1 requires trading strategies to be optimal for arbitrary trading histories, one must consider the additional terms. In particular, equilibrium parameters will depend on off-equilibrium parameters as the following proposition shows.

Proposition 1.1. *The strategies and beliefs (1.8), (1.9), (1.10), (1.22), (1.23), (1.24), and (1.25) form a symmetric linear Markov perfect equilibrium if the following recursions hold for all periods $n \in \{1, \ldots, N\}$:*

$$\beta_n = \frac{\eta_n - \lambda_n\psi_n - (1-\zeta_n\psi_n)(e_n\lambda_n + f_n\zeta_n + g_n\theta_n)}{\lambda_n[2+(M-1)\phi_n] - \zeta_n[1+(M-1)\phi_n](e_n\lambda_n + f_n\zeta_n + g_n\theta_n)},$$

$$\gamma_n = \frac{1-2b_n\lambda_n - h_n\zeta_n - i_n\theta_n}{\lambda_n(2-2b_n\lambda_n - h_n\zeta_n - i_n\theta_n) - \zeta_n(2c_n\zeta_n + h_n\lambda_n + j_n\theta_n) - \theta_n(2d_n\theta_n + i_n\lambda_n + j_n\zeta_n)},$$

$$\alpha_n = \frac{\begin{array}{c}(M-1)\beta_n\Big[\theta_n(2d_n\theta_n + i_n\lambda_n + j_n\zeta_n) - \lambda_n(1-2b_n\lambda_n - h_n\zeta_n - i_n\theta_n)\Big] \\ -\,[1-(M-1)\beta_n\zeta_n](2c_n\zeta_n + h_n\lambda_n + j_n\theta_n)\end{array}}{\lambda_n(2-2b_n\lambda_n - h_n\zeta_n - i_n\theta_n) - \zeta_n(2c_n\zeta_n + h_n\lambda_n + j_n\theta_n) - \theta_n(2d_n\theta_n + i_n\lambda_n + j_n\zeta_n)},$$

$$\delta_n = \frac{\zeta_n(2c_n\zeta_n + h_n\lambda_n + j_n\theta_n) - \lambda_n(1-2b_n\lambda_n - h_n\zeta_n - i_n\theta_n) - (1-\theta_n)(2d_n\theta_n + i_n\lambda_n + j_n\zeta_n)}{\lambda_n(2-2b_n\lambda_n - h_n\zeta_n - i_n\theta_n) - \zeta_n(2c_n\zeta_n + h_n\lambda_n + j_n\theta_n) - \theta_n(2d_n\theta_n + i_n\lambda_n + j_n\zeta_n)},$$

$$\lambda_n = \frac{M\beta_n\Sigma_{n-1}^{v,s_{i,0}} + \Sigma_{n-1}^{v,u_n}}{M\beta_n^2\Big[\Sigma_{n-1}^{s_{i,0}} + (M-1)\Sigma_{n-1}^{s_{i,0},s_{j,0}}\Big] + \Sigma_{n-1}^{u_n} + 2M\beta_n\Sigma_{n-1}^{s_{i,0},u_n}},$$

$$\zeta_n = \frac{\beta_n\Big[\Sigma_{n-1}^{s_{i,0}} + (M-1)\Sigma_{n-1}^{s_{i,0},s_{j,0}}\Big] + \Sigma_{n-1}^{s_{i,0},u_n}}{M\beta_n^2\Big[\Sigma_{n-1}^{s_{i,0}} + (M-1)\Sigma_{n-1}^{s_{i,0},s_{j,0}}\Big] + \Sigma_{n-1}^{u_n} + 2M\beta_n\Sigma_{n-1}^{s_{i,0},u_n}},$$

$$\theta_n = \frac{M\beta_n\Sigma_{n-1}^{s_{i,0},u_n} + \Sigma_{n-1}^{u_n,u_{n+1}}}{M\beta_n^2\Big[\Sigma_{n-1}^{s_{i,0}} + (M-1)\Sigma_{n-1}^{s_{i,0},s_{j,0}}\Big] + \Sigma_{n-1}^{u_n} + 2M\beta_n\Sigma_{n-1}^{s_{i,0},u_n}},$$

with value function coefficients

$$a_{n-1} = \beta_n\Big(\eta_n - \lambda_n\Big[\beta_n + (M-1)\beta_n\phi_n + \psi_n\Big]\Big) + a_n\Big(1-\zeta_n\Big[\beta_n + (M-1)\beta_n\phi_n + \psi_n\Big]\Big)^2,$$

$$b_{n-1} = b_n(1-\gamma_n\lambda_n)^2 + \gamma_n(1-\gamma_n\lambda_n)(1-h_n\zeta_n - i_n\theta_n) + \gamma_n^2(c_n\zeta_n^2 + d_n\theta_n^2 + j_n\zeta_n\theta_n),$$

$$\begin{aligned} c_{n-1} = {} & c_n\Big(1-\zeta_n\Big[\alpha_n + (M-1)\beta_n\Big]\Big)^2 + \Big[\alpha_n + (M-1)\beta_n\Big]^2(b_n\lambda_n^2 + d_n\theta_n^2 + i_n\lambda_n\theta_n) \\ & - \Big[\alpha_n + (M-1)\beta_n\Big]\Big[\alpha_n\lambda_n + \Big(1-\zeta_n\Big[\alpha_n + (M-1)\beta_n\Big]\Big)(h_n\lambda_n + j_n\theta_n)\Big], \end{aligned}$$

$$\begin{aligned} d_{n-1} = {} & d_n\Big(1-\theta_n[1+\delta_n]\Big)^2 + (1+\delta_n)^2(b_n\lambda_n^2 + c_n\zeta_n^2 + h_n\lambda_n\zeta_n) \\ & - (1+\delta_n)\Big[\delta_n\lambda_n + \Big(1-\theta_n[1+\delta_n]\Big)(i_n\lambda_n + j_n\zeta_n)\Big], \end{aligned}$$

$$\begin{aligned} e_{n-1} = {} & \Big[e_n(1-\gamma_n\lambda_n) - \gamma_n(f_n\zeta_n + g_n\theta_n)\Big]\Big(1-\zeta_n\Big[\beta_n + (M-1)\beta_n\phi_n + \psi_n\Big]\Big) \\ & + \gamma_n\Big(\eta_n - \lambda_n\Big[\beta_n + (M-1)\beta_n\phi_n + \psi_n\Big]\Big) + \beta_n(1-\gamma_n\lambda_n), \end{aligned}$$

$$
\begin{aligned}
f_{n-1} = &\Big[f_n\Big(1-\zeta_n\Big[\alpha_n+(M-1)\beta_n\Big]\Big)-\Big[\alpha_n+(M-1)\beta_n\Big](e_n\lambda_n+g_n\theta_n)\Big]\Big(1-\zeta_n\Big[\beta_n\\
&+(M-1)\beta_n\phi_n+\psi_n\Big]\Big)+\alpha_n\Big(\eta_n-\lambda_n\Big[\beta_n+(M-1)\beta_n\phi_n+\psi_n\Big]\Big)\\
&-\lambda_n\beta_n\Big[\alpha_n+(M-1)\beta_n\Big],
\end{aligned}
$$

$$
\begin{aligned}
g_{n-1} = &\Big[g_n\Big(1-\theta_n[1+\delta_n]\Big)-(1+\delta_n)(e_n\lambda_n+f_n\zeta_n)\Big]\Big(1-\zeta_n\Big[\beta_n+(M-1)\beta_n\phi_n+\psi_n\Big]\Big)\\
&+\delta_n\Big(\eta_n-\lambda_n\Big[\beta_n+(M-1)\beta_n\phi_n+\psi_n\Big]\Big)-\lambda_n\beta_n(1+\delta_n),
\end{aligned}
$$

$$
\begin{aligned}
h_{n-1} = &\Big[h_n(1-\gamma_n\lambda_n)-\gamma_n(2c_n\zeta_n+j_n\theta_n)\Big]\Big(1-\zeta_n\Big[\alpha_n+(M-1)\beta_n\Big]\Big)+\alpha_n(1-\gamma_n\lambda_n)\\
&+\Big[\alpha_n+(M-1)\beta_n\Big]\Big[\gamma_n\Big[\theta_n(2d_n\theta_n+j_n\zeta_n)-\lambda_n(1-h_n\zeta_n-i_n\theta_n)\Big]\\
&-(1-\gamma_n\lambda_n)(2b_n\lambda_n+i_n\theta_n)\Big],
\end{aligned}
$$

$$
\begin{aligned}
i_{n-1} = &\Big[i_n(1-\gamma_n\lambda_n)-\gamma_n(2d_n\theta_n+j_n\zeta_n)\Big]\Big(1-\theta_n[1+\delta_n]\Big)+\delta_n(1-\gamma_n\lambda_n)\\
&+(1+\delta_n)\Big[\gamma_n\Big[\zeta_n(2c_n\zeta_n+j_n\theta_n)-\lambda_n(1-h_n\zeta_n-i_n\theta_n)\Big]-(1-\gamma_n\lambda_n)(2b_n\lambda_n+h_n\zeta_n)\Big],
\end{aligned}
$$

$$
\begin{aligned}
j_{n-1} = &\Big[j_n\Big(1-\theta_n[1+\delta_n]\Big)-(1+\delta_n)(2c_n\zeta_n+h_n\lambda_n)\Big]\Big(1-\zeta_n\Big[\alpha_n+(M-1)\beta_n\Big]\Big)\\
&-\alpha_n\lambda_n(1+\delta_n)+\Big[\alpha_n+(M-1)\beta_n\Big]\Big[(1+\delta_n)\Big[\lambda_n(2b_n\lambda_n+h_n\zeta_n)+\theta_n(i_n\lambda_n+j_n\zeta_n)\Big]\\
&-\lambda_n\delta_n-\Big(1-\theta_n[1+\delta_n]\Big)(2d_n\theta_n+i_n\lambda_n)\Big],
\end{aligned}
$$

$$
\begin{aligned}
k_{n-1} = &\,k_n+a_n\zeta_n^2\Big[\operatorname{Var}\Big[u_n-r_{n,n-1}\Big|s_{i,0},y_{1:n-1}\Big]+2(M-1)\beta_n\operatorname{Cov}\Big[s_{j,n-1},u_n-r_{n,n-1}\Big|s_{i,0},y_{1:n-1}\Big]\\
&+(M-1)\beta_n^2\Big(\operatorname{Var}\Big[s_{j,n-1}\Big|s_{i,0},y_{1:n-1}\Big]+(M-2)\operatorname{Cov}\Big[s_{j,n-1},s_{k,n-1}\Big|s_{i,0},y_{1:n-1}\Big]\Big)\Big],
\end{aligned}
$$

where $a_N=b_N=c_N=d_N=e_N=f_N=g_N=h_N=i_N=j_N=k_N=0$ *and*

$$
\eta_n=\frac{\Sigma_{n-1}^{v,s_{i,0}}}{\Sigma_{n-1}^{s_{i,0}}},
$$

$$
\phi_n=\frac{\Sigma_{n-1}^{s_{i,0},s_{j,0}}}{\Sigma_{n-1}^{s_{i,0}}},
$$

$$
\psi_n=\frac{\Sigma_{n-1}^{s_{i,0},u_n}}{\Sigma_{n-1}^{s_{i,0}}}.
$$

Also, for all $i, j, k \in \{1, \ldots, M\}$ where $i \neq j \neq k$:

$$\mathrm{Var}\left[u_n - r_{n,n-1} \middle| s_{i,0}, y_{1:n-1}\right] = \Sigma_{n-1}^{u_n} - \psi_n^2 \Sigma_{n-1}^{s_{i,0}},$$

$$\mathrm{Var}\left[s_{j,n-1} \middle| s_{i,0}, y_{1:n-1}\right] = (1 - \phi_n^2) \Sigma_{n-1}^{s_{i,0}},$$

$$\mathrm{Cov}\left[s_{j,n-1}, s_{k,n-1} \middle| s_{i,0}, y_{1:n-1}\right] = (1 - \phi_n) \Sigma_{n-1}^{s_{i,0}, s_{j,0}},$$

$$\mathrm{Cov}\left[s_{j,n-1}, u_n - r_{n,n-1} \middle| s_{i,0}, y_{1:n-1}\right] = (1 - \phi_n) \Sigma_{n-1}^{s_{i,0}, u_n}.$$

Moreover, the second-order condition must hold in every period:

$$\lambda_n(2 - 2b_n\lambda_n - h_n\zeta_n - i_n\theta_n) - \zeta_n(2c_n\zeta_n + h_n\lambda_n + j_n\theta_n) - \theta_n(2d_n\theta_n + i_n\lambda_n + j_n\zeta_n) > 0.$$

Finally, conditional (co)variances must satisfy the following recursive block structure in every period:

$$\Sigma_n^{v} = \Sigma_{n-1}^{v} - \lambda_n^2\Big(M\beta_n^2\Big[\Sigma_{n-1}^{s_{i,0}} + (M-1)\Sigma_{n-1}^{s_{i,0},s_{j,0}}\Big] + \Sigma_{n-1}^{u_n} + 2M\beta_n\Sigma_{n-1}^{s_{i,0},u_n}\Big),$$

$$\Sigma_n^{v,s_{i,0}} = \Sigma_{n-1}^{v,s_{i,0}} - \lambda_n\zeta_n\Big(M\beta_n^2\Big[\Sigma_{n-1}^{s_{i,0}} + (M-1)\Sigma_{n-1}^{s_{i,0},s_{j,0}}\Big] + \Sigma_{n-1}^{u_n} + 2M\beta_n\Sigma_{n-1}^{s_{i,0},u_n}\Big),$$

$$\Sigma_n^{v,u_{n+1}} = \Sigma_{n-1}^{v,u_n} - \lambda_n\theta_n\Big(M\beta_n^2\Big[\Sigma_{n-1}^{s_{i,0}} + (M-1)\Sigma_{n-1}^{s_{i,0},s_{j,0}}\Big] + \Sigma_{n-1}^{u_n} + 2M\beta_n\Sigma_{n-1}^{s_{i,0},u_n}\Big),$$

$$\Sigma_n^{s_{i,0}} = \Sigma_{n-1}^{s_{i,0}} - \zeta_n^2\Big(M\beta_n^2\Big[\Sigma_{n-1}^{s_{i,0}} + (M-1)\Sigma_{n-1}^{s_{i,0},s_{j,0}}\Big] + \Sigma_{n-1}^{u_n} + 2M\beta_n\Sigma_{n-1}^{s_{i,0},u_n}\Big),$$

$$\Sigma_n^{s_{i,0},s_{j,0}} = \Sigma_{n-1}^{s_{i,0},s_{j,0}} - \zeta_n^2\Big(M\beta_n^2\Big[\Sigma_{n-1}^{s_{i,0}} + (M-1)\Sigma_{n-1}^{s_{i,0},s_{j,0}}\Big] + \Sigma_{n-1}^{u_n} + 2M\beta_n\Sigma_{n-1}^{s_{i,0},u_n}\Big),$$

$$\Sigma_n^{s_{i,0},u_{n+1}} = \Sigma_{n-1}^{s_{i,0},u_n} - \zeta_n\theta_n\Big(M\beta_n^2\Big[\Sigma_{n-1}^{s_{i,0}} + (M-1)\Sigma_{n-1}^{s_{i,0},s_{j,0}}\Big] + \Sigma_{n-1}^{u_n} + 2M\beta_n\Sigma_{n-1}^{s_{i,0},u_n}\Big),$$

$$\Sigma_n^{u_{n+1}} = \Sigma_{n-1}^{u_n} - \theta_n^2\Big(M\beta_n^2\Big[\Sigma_{n-1}^{s_{i,0}} + (M-1)\Sigma_{n-1}^{s_{i,0},s_{j,0}}\Big] + \Sigma_{n-1}^{u_n} + 2M\beta_n\Sigma_{n-1}^{s_{i,0},u_n}\Big),$$

$$\Sigma_n^{u_{n+1},u_{n+2}} = \Sigma_{n-1}^{u_n,u_{n+1}} - \theta_n^2\Big(M\beta_n^2\Big[\Sigma_{n-1}^{s_{i,0}} + (M-1)\Sigma_{n-1}^{s_{i,0},s_{j,0}}\Big] + \Sigma_{n-1}^{u_n} + 2M\beta_n\Sigma_{n-1}^{s_{i,0},u_n}\Big).$$

Proof. See Appendix 1.A.6.

Proposition 1.1 specifies the necessary and sufficient conditions for a dynamic trading equilibrium. As is the case with most discrete-time strategic trading models, including Foster and Viswanathan (1996), this paper does not have analytic expressions for the equilibrium. Therefore, Proposition 1.1 must be computed numerically. The corresponding algorithm is described in Appendix 1.A.7.

While Σ_n^v is not technically required to solve the remaining system of equations, one still needs to keep track of it for the numerical analysis. Since the equilibrium is only sustainable if the market maker does not fully learn about the asset's true value before the final trading period, one has to ensure that $\Sigma_{N-1}^v > 0$.

Moreover, from the equations for θ_n, $\Sigma_n^{v,u_{n+1}}$, $\Sigma_n^{s_{i,0},u_{n+1}}$, and $\Sigma_n^{u_{n+1},u_{n+2}}$, it can be inferred that uninformed orders are conditionally correlated with the asset value and informed traders' signals in the cross-section if they are autocorrelated in the time series. Thus, even if autocorrelated uninformed orders do not initially contain information about the asset value and signals, they do so upon conditioning. Similarly, if uninformed orders are initially uncorrelated over time, they become conditionally autocorrelated as long as they initially contain information about informed traders' signals.

Finally, note that the models by Kyle (1985), Holden and Subrahmanyam (1992), and Foster and Viswanathan (1996) are all nested in Proposition 1.1. If one assumes uninformed orders to be white noise, one obtains the Foster and Viswanathan (1996) model. Additionally, if informed traders perfectly observe the asset value v, one gets the Holden and Subrahmanyam (1992) model with homogeneously informed traders, while the Kyle (1985) model is obtained for the special case where there is only one informed trader.

1.5 Numerical Analysis

This study is interested in how (auto)correlated uninformed orders affect the equilibrium outcome. Numerically, correlations have to be bounded above in absolute value to ensure that the covariance matrix $\mathbf{\Sigma_0}$ in (1.1) is positive definite. Denoting $\rho_0^{\bar{x},\bar{y}}$ as the unconditional correlation coefficient between two arbitrary variables $\bar{x}$ and $\bar{y}$, the absolute values of $\rho_0^{v,s_{i,0}}$, ρ_0^{v,u_n}, $\rho_0^{s_{i,0},u_n}$, $\rho_0^{s_{i,0},s_{j,0}}$, and $\rho_0^{u_n,u_{n+1}}$

have to be bounded above for the model to be well-defined. Therefore, the equilibrium in Proposition 1.1 is only sustainable for small to moderate correlations.

The main variables of interest are ρ_0^{v,u_n}, the unconditional correlation between the asset value and uninformed orders, and $\rho_0^{s_{i,0},u_n}$, the unconditional correlation between informed traders' signals and uninformed orders. This study fixes the number of informed traders to $M = 3$ and the number of trading periods to $N = 10$. Initial variances equal $\Sigma_0^v = 5$, $\Sigma_0^{s_{i,0}} = 1$, and $\Sigma_0^{u_n} = 3/N$. The unconditional signal informativeness equals $\rho_0^{v,s_{i,0}} = 0.4$, and the unconditional autocorrelation in the uninformed order flow is set to $\rho_0^{u_n,u_{n+1}} = 0.1$ since positive autocorrelation is empirically the most relevant case.[10] Finally, three different unconditional signal correlations are considered: positive correlation $\rho_0^{s_{i,0},s_{j,0}} = 0.25$, zero correlation $\rho_0^{s_{i,0},s_{j,0}} = 0$, and negative correlation $\rho_0^{s_{i,0},s_{j,0}} = -0.25$. Different combinations of ρ_0^{v,u_n} and $\rho_0^{s_{i,0},u_n}$ are analyzed for each unconditional signal correlation.

1.5.1 Positive Initial Signal Correlation

In this analysis, the initial signal correlation equals $\rho_0^{s_{i,0},s_{j,0}} = 0.25$. For uninformed orders, the unconditional correlation structures are ($\rho_0^{v,u_n} = 0.25$, $\rho_0^{s_{i,0},u_{n,0}} = 0.25$), ($\rho_0^{v,u_n} = 0.1$, $\rho_0^{s_{i,0},u_n} = -0.1$), ($\rho_0^{v,u_n} = -0.1$, $\rho_0^{s_{i,0},u_n} = 0.1$), and ($\rho_0^{v,u_n} = -0.25$, $\rho_0^{s_{i,0},u_n} = -0.25$).

Figure 1.1 plots informed traders' trading intensity β_n, the market maker's illiquidity parameter λ_n, the conditional asset variance Σ_n^v, and informed traders' terminal conditional expected profits over time. Suppose uninformed orders are initially positively correlated with the asset value and signals. In that case, the market maker's price sensitivity monotonically decreases over time since most information about the asset value gets incorporated into prices in the first periods. Since uninformed orders reveal information about the asset's true value, competition for informed traders is high. Consequently, informed traders make the lowest expected profits and decrease trading intensities throughout the trading day.

[10] See Hasbrouck (1991a,b) and Brogaard, Hendershott, and Riordan (2019).

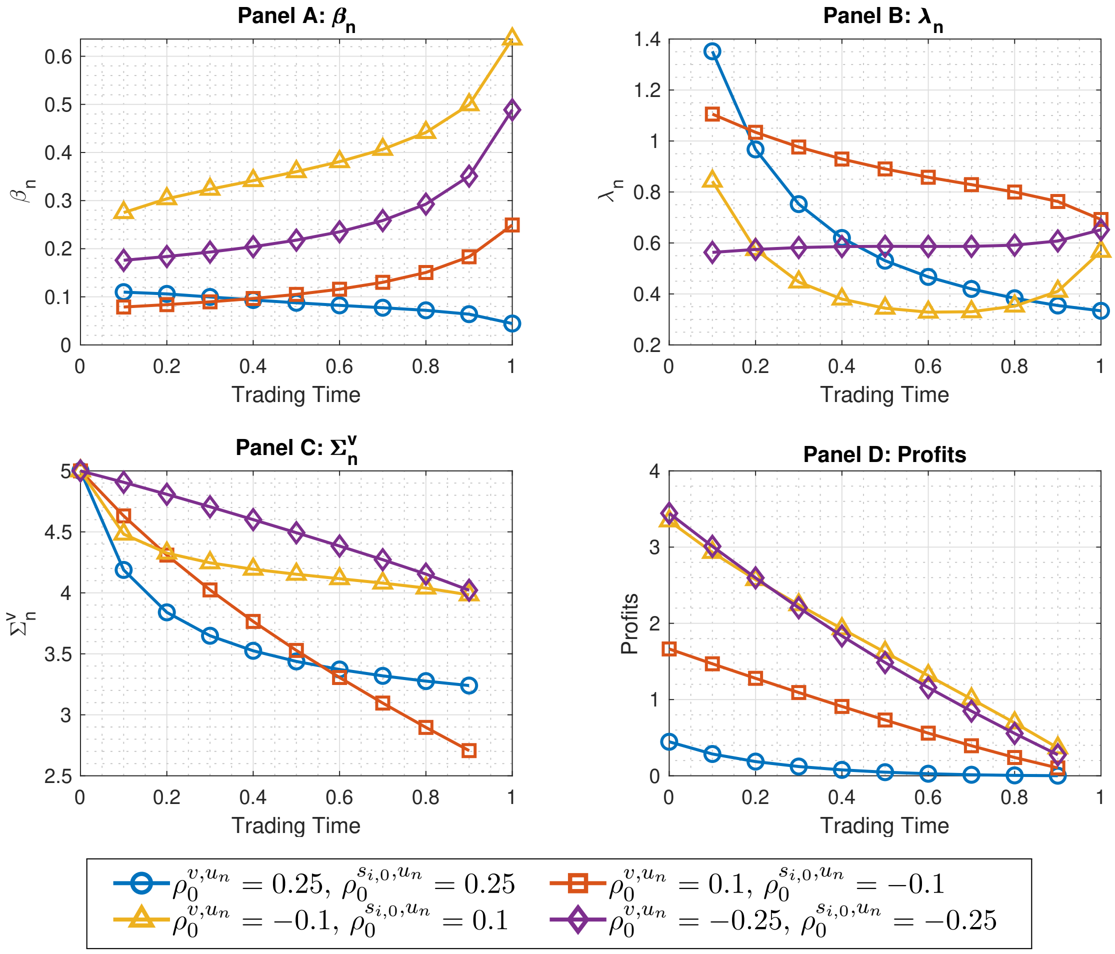

Figure 1.1: Trading Intensity, Illiquidity Parameter, Conditional Asset Variance, and Expected Profits with Positive Initial Signal Correlation. The figure plots informed traders' trading intensity β_n, the market maker's illiquidity parameter λ_n, the conditional asset variance Σ_n^v, and informed traders' terminal conditional expected profits over time. Parameter values are $M = 3$, $N = 10$, $\Sigma_0^v = 5$, $\Sigma_0^{s_{i,0}} = 1$, $\Sigma_0^{u_n} = 3/N$, $\rho_0^{v,s_{i,0}} = 0.4$, $\rho_0^{s_{i,0},s_{j,0}} = 0.25$, and $\rho_0^{u_n,u_{n+1}} = 0.1$. The model is solved for four initial parametrizations: $(\rho_0^{v,u_n} = 0.25,\ \rho_0^{s_{i,0},u_n} = 0.25)$, $(\rho_0^{v,u_n} = 0.1,\ \rho_0^{s_{i,0},u_n} = -0.1)$, $(\rho_0^{v,u_n} = -0.1,\ \rho_0^{s_{i,0},u_n} = 0.1)$, and $(\rho_0^{v,u_n} = -0.25,\ \rho_0^{s_{i,0},u_n} = -0.25)$.

If uninformed orders are initially positively correlated with the asset value but negatively correlated with signals, informed traders' trading intensity monotonically increases over time, even though uninformed traders still reveal information about the true asset value. Surprisingly, at the end of the trading day, price discovery is the highest. In that scenario, aggregate order flows remain consistently informative about the asset value, resulting in most information getting incorporated into prices. Consequently, the market maker's price sensitivity only slowly decreases over time.

Suppose uninformed orders are initially negatively correlated with the asset value but positively correlated with signals. In that case, uninformed traders push the price away from the true asset value, which provides camouflage for informed traders. As a result, informed traders trade most aggressively on their residual signals throughout the trading day and make high expected profits. Interestingly, the market maker's illiquidity parameter exhibits a clear U-shape. It initially declines because aggregate order flows are most informative about the asset value in the first trading rounds. However, the illiquidity parameter eventually increases in later periods since informed traders submit their most aggressive orders towards the end of the trading day.

Finally, if uninformed orders are initially negatively correlated with the asset value and signals, informed traders also increase their trading intensity over time and make high expected profits. As in Kyle (1985), informed traders have an incentive to delay their trades towards later periods to capitalize on their private information. Moreover, the market maker can extract the least information from aggregate order flows since uninformed traders trade in the opposite direction of the true asset value and signals. Consequently, the market maker keeps a steady price sensitivity, resulting in slow price discovery over time.

Figure 1.2 shows the evolution of the conditional correlations $\rho_n^{u_{n+1},u_{n+2}}$, $\rho_n^{s_{i,0},s_{j,0}}$, $\rho_n^{v,u_{n+1}}$, and $\rho_n^{s_{i,0},u_{n+1}}$. If uninformed orders are initially positively correlated with the asset value and signals, the conditional correlations monotonically decrease over time. Thus, uninformed orders become less informative about the asset value and signals in later periods. Furthermore, $\rho_n^{u_{n+1},u_{n+2}}$ strongly decreases in the first periods so that uninformed orders become conditionally less autocorrelated. Finally, $\rho_n^{s_{i,0},s_{j,0}}$ becomes negative over time, meaning that informed traders switch from trading in the same direction to trading in opposite directions. This result is equivalent to informed traders developing a difference of opinion about the true asset value.

Suppose uninformed orders are initially positively correlated with the asset value but negatively correlated with signals. In that case, the conditional signal corre-

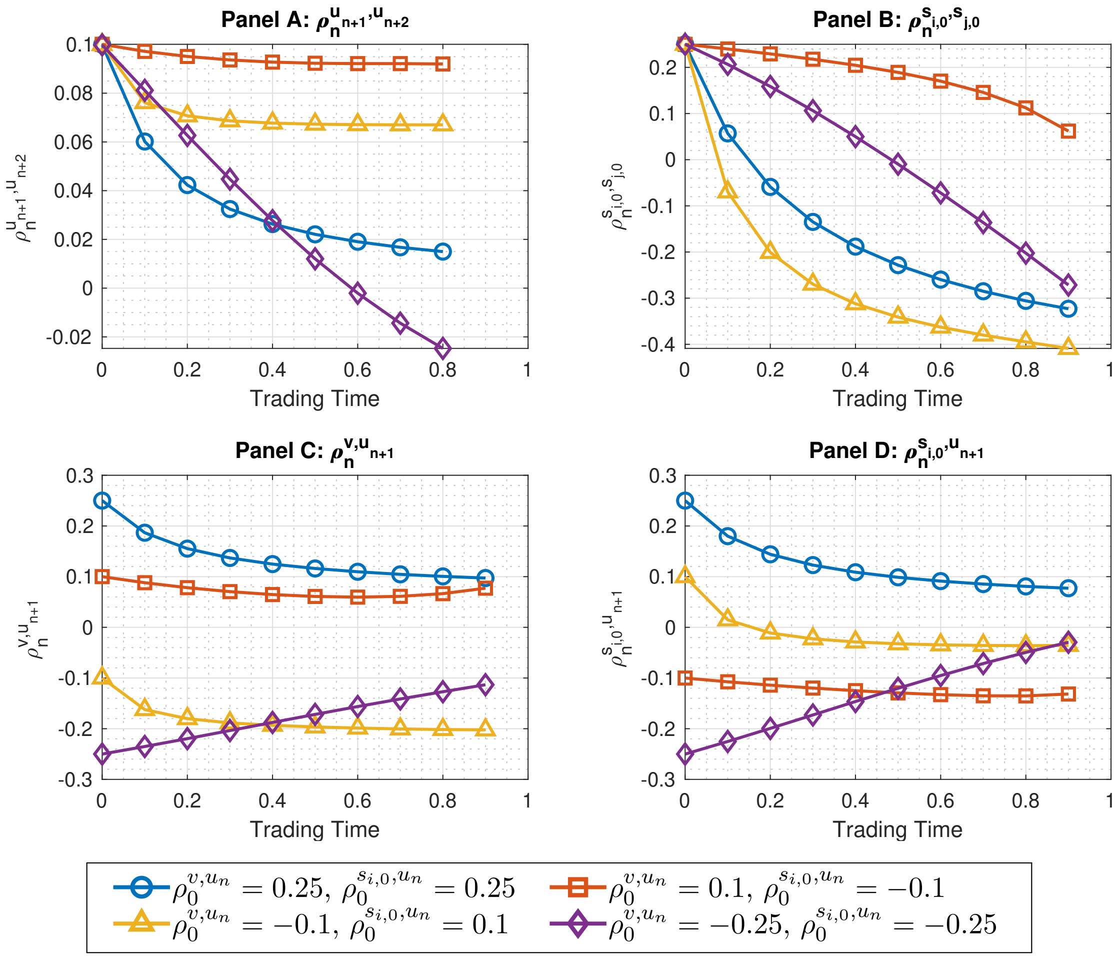

Figure 1.2: Conditional Correlations with Positive Initial Signal Correlation. The figure plots the conditional correlation between future uninformed orders $\rho_n^{u_{n+1},u_{n+2}}$, the conditional correlation between informed traders' signals $\rho_n^{s_{i,0},s_{j,0}}$, the conditional correlation between the asset value and future uninformed orders $\rho_n^{v,u_{n+1}}$, and the conditional correlation between informed traders' signals and future uninformed orders $\rho_n^{s_{i,0},u_{n+1}}$ over time. Parameter values are $M = 3$, $N = 10$, $\Sigma_0^v = 5$, $\Sigma_0^{s_{i,0}} = 1$, $\Sigma_0^{u_n} = 3/N$, $\rho_0^{v,s_{i,0}} = 0.4$, $\rho_0^{s_{i,0},s_{j,0}} = 0.25$, and $\rho_0^{u_n,u_{n+1}} = 0.1$. The model is solved for four initial parametrizations: ($\rho_0^{v,u_n} = 0.25$, $\rho_0^{s_{i,0},u_n} = 0.25$), ($\rho_0^{v,u_n} = 0.1$, $\rho_0^{s_{i,0},u_n} = -0.1$), ($\rho_0^{v,u_n} = -0.1$, $\rho_0^{s_{i,0},u_n} = 0.1$), and ($\rho_0^{v,u_n} = -0.25$, $\rho_0^{s_{i,0},u_n} = -0.25$).

lation slowly decreases, while the remaining conditional correlations stay approximately constant over time. Therefore, uninformed orders remain consistently informative about the asset value and exhibit a persistently high conditional autocorrelation. Moreover, the conditional signal correlation stays positive throughout the whole trading day so that no switch in trading directions occurs. Finally, note that $\rho_n^{v,u_{n+1}}$ is U-shaped and thus non-monotonic.

If uninformed orders are initially negatively correlated with the asset value but positively correlated with signals, the conditional correlations monotonically decline over time. In that scenario, uninformed traders effectively conceal informed traders' orders from the market maker by pushing the price further away from the asset's true value while losing their informational value about signals after the first few trading rounds. Furthermore, the conditional signal correlation becomes negative after the first period. Thus, even if informed traders initially agree whether the asset is overvalued or undervalued, they will have different opinions after only one round of trading.

Finally, if uninformed orders are initially negatively correlated with the asset value and signals, $\rho_n^{u_{n+1},u_{n+2}}$ and $\rho_n^{s_{i,0},s_{j,0}}$ monotonically decrease, while $\rho_n^{v,u_{n+1}}$ and $\rho_n^{s_{i,0},u_{n+1}}$ monotonically increase over time. Since uninformed orders become less negatively correlated with the asset value and signals, uninformed traders provide less camouflage for informed traders in later periods. Surprisingly, the conditional uninformed order flow autocorrelation becomes negative over time. Thus, even if uninformed orders are positively autocorrelated, they can become negatively autocorrelated upon conditioning.

1.5.2 Zero Initial Signal Correlation

In this analysis, the initial signal correlation equals $\rho_0^{s_{i,0},s_{j,0}} = 0$. For uninformed orders, the unconditional correlation structures are ($\rho_0^{v,u_n} = 0.2$, $\rho_0^{s_{i,0},u_n} = 0.1$), ($\rho_0^{v,u_n} = 0.1$, $\rho_0^{s_{i,0},u_n} = -0.1$), ($\rho_0^{v,u_n} = -0.1$, $\rho_0^{s_{i,0},u_n} = 0.1$), and ($\rho_0^{v,u_n} = -0.1$, $\rho_0^{s_{i,0},u_n} = -0.2$).

Figure 1.3 plots informed traders' trading intensity β_n, the market maker's illiquidity parameter λ_n, the conditional asset variance Σ_n^v, and informed traders' terminal conditional expected profits over time. It is worth noting that, except for ($\rho_0^{v,u_n} = -0.1$, $\rho_0^{s_{i,0},u_n} = 0.1$), trading intensities are approximately unaffected by the correlation structure of uninformed orders if informed traders receive independent signals. Also, trading intensities monotonically increase over time.

If uninformed orders are initially positively correlated with the asset value and signals, informed traders make the lowest expected profits. Since uninformed traders reveal information about the true asset value, competition for informed

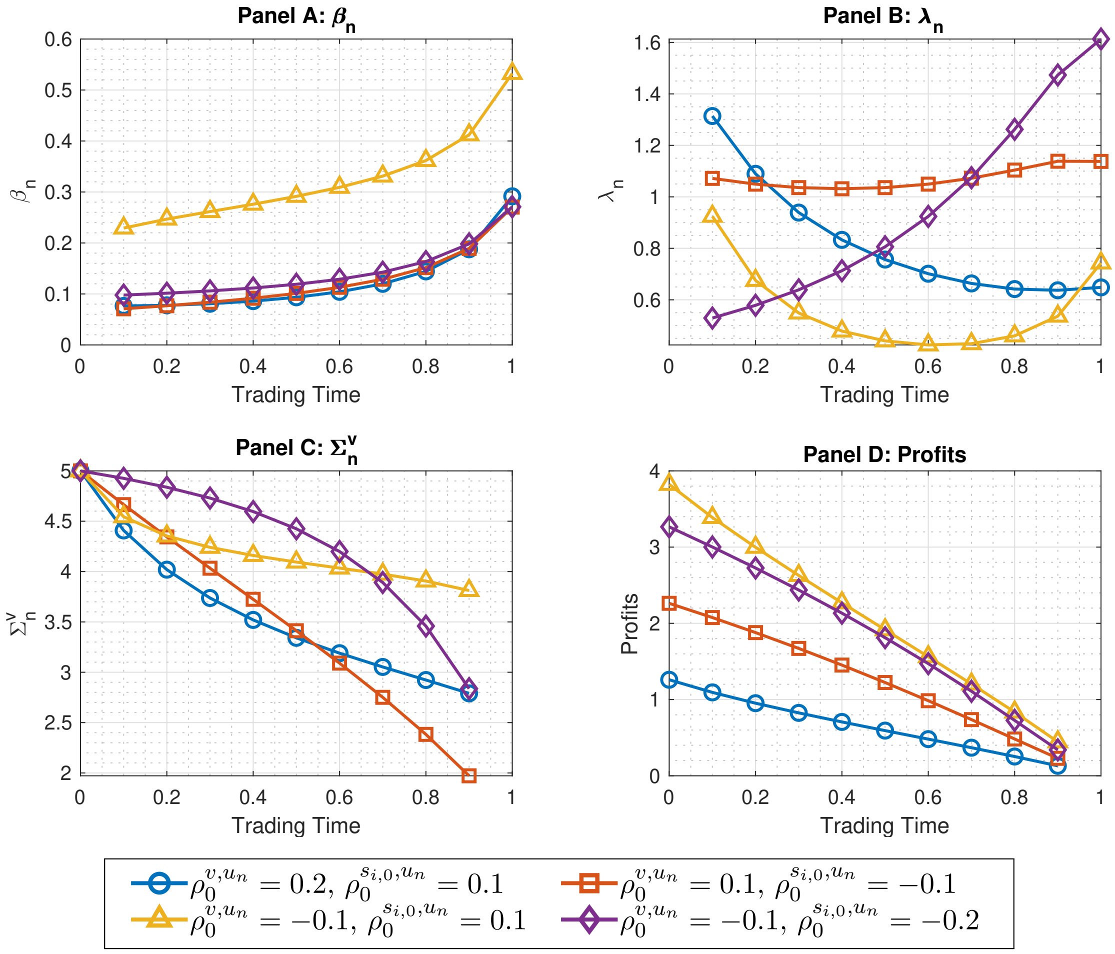

Figure 1.3: Trading Intensity, Illiquidity Parameter, Conditional Asset Variance, and Expected Profits with Zero Initial Signal Correlation. The figure plots informed traders' trading intensity β_n, the market maker's illiquidity parameter λ_n, the conditional asset variance Σ_n^v, and informed traders' terminal conditional expected profits over time. Parameter values are $M = 3$, $N = 10$, $\Sigma_0^v = 5$, $\Sigma_0^{s_{i,0}} = 1$, $\Sigma_0^{u_n} = 3/N$, $\rho_0^{v,s_{i,0}} = 0.4$, $\rho_0^{s_{i,0},s_{j,0}} = 0$, and $\rho_0^{u_n,u_{n+1}} = 0.1$. The model is solved for four initial parametrizations: ($\rho_0^{v,u_n} = 0.2$, $\rho_0^{s_{i,0},u_n} = 0.1$), ($\rho_0^{v,u_n} = 0.1$, $\rho_0^{s_{i,0},u_n} = -0.1$), ($\rho_0^{v,u_n} = -0.1$, $\rho_0^{s_{i,0},u_n} = 0.1$), and ($\rho_0^{v,u_n} = -0.1$, $\rho_0^{s_{i,0},u_n} = -0.2$).

traders is high. Moreover, the market maker's price sensitivity monotonically decreases since aggregate order flows become less informative about the asset value over time. Consequently, price discovery is highest in the first trading periods.

If uninformed orders are initially positively correlated with the asset value but negatively correlated with signals, informed traders make moderate expected profits. Simultaneously, the market maker's price sensitivity remains approximately constant throughout the trading day. In that scenario, aggregate order flows stay consistently informative about the asset value, resulting in the highest price discovery at the end of the trading day.

If uninformed orders are initially negatively correlated with the asset value but positively correlated with signals, informed traders trade most aggressively on their residual signals and make high expected profits. In that scenario, uninformed traders push the price away from the true asset value while not being informative about signals after the first few trading rounds. As a result, price discovery is lowest at the end of the trading day since informed traders can effectively conceal their information from the market maker. Moreover, the illiquidity parameter exhibits a clear U-shape. It initially declines because the first aggregate order flows contain the most information about the asset value. However, the illiquidity parameter eventually increases in later trading rounds since informed traders trade most aggressively on their private information towards the end of the trading day.

Finally, if uninformed orders are initially negatively correlated with the asset value and signals, informed traders also make high expected profits. Furthermore, the illiquidity parameter monotonically increases over time. In that scenario, uninformed orders provide camouflage for informed traders by trading in the opposite direction of the true asset value and signals. As a result, the market maker increases her price sensitivity throughout the trading day since she can extract the least information from aggregate order flows in the first periods.

Figure 1.4 shows the evolution of the conditional correlations $\rho_n^{u_{n+1},u_{n+2}}$, $\rho_n^{s_{i,0},s_{j,0}}$, $\rho_n^{v,u_{n+1}}$, and $\rho_n^{s_{i,0},u_{n+1}}$. If uninformed orders are initially positively correlated with the asset value and signals, the conditional correlations monotonically decrease over time. Therefore, uninformed orders become less informative about the asset value and signals in later periods. Moreover, informed traders develop a difference of opinion about the asset value after only one round of trading, and the conditional uninformed order flow autocorrelation strongly declines over time.

Suppose uninformed orders are initially positively correlated with the asset value but negatively correlated with signals. In that case, the conditional uninformed order flow autocorrelation slowly decreases, while the remaining conditional correlations stay approximately constant over time. Since aggregate order flows remain

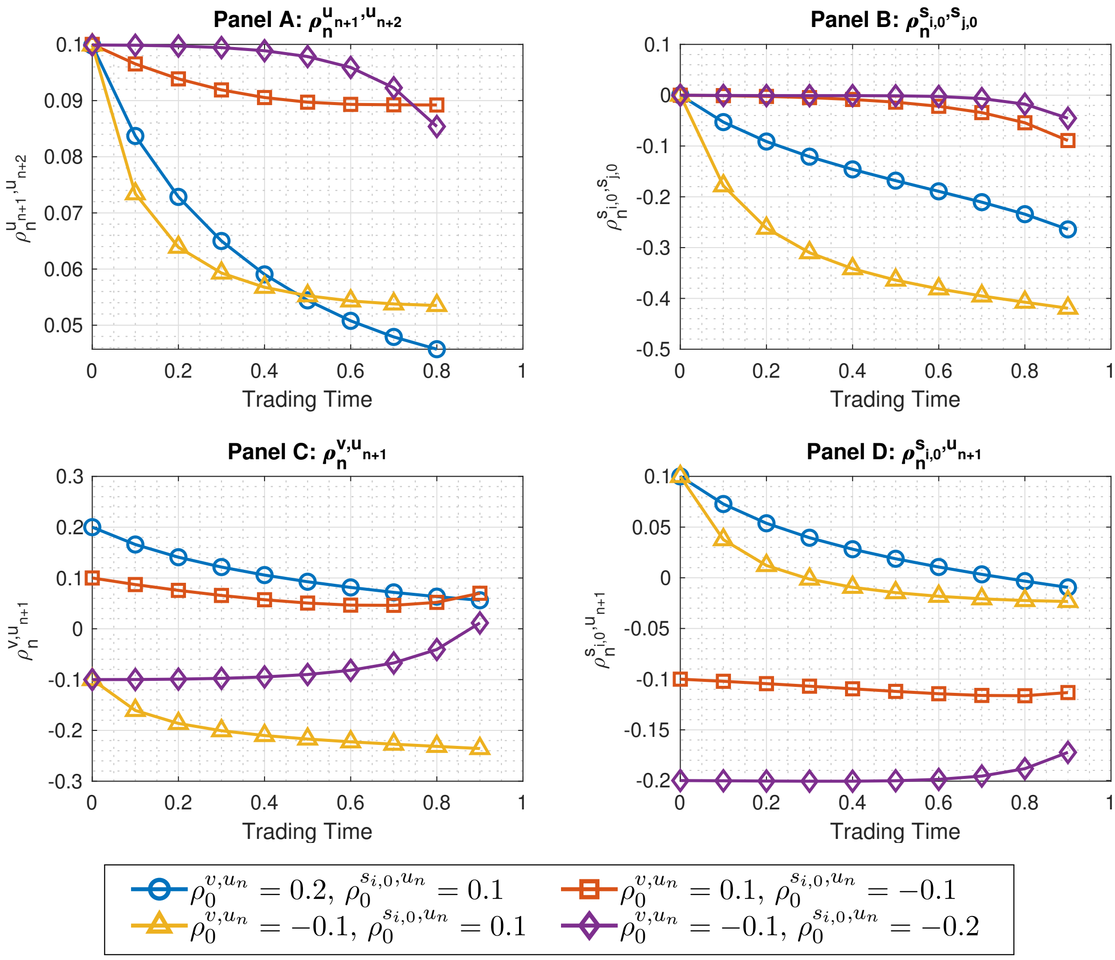

Figure 1.4: Conditional Correlations with Zero Initial Signal Correlation. The figure plots the conditional correlation between future uninformed orders $\rho_n^{u_{n+1},u_{n+2}}$, the conditional correlation between informed traders' signals $\rho_n^{s_{i,0},s_{j,0}}$, the conditional correlation between the asset value and future uninformed orders $\rho_n^{v,u_{n+1}}$, and the conditional correlation between informed traders' signals and future uninformed orders $\rho_n^{s_{i,0},u_{n+1}}$ over time. Parameter values are $M = 3$, $N = 10$, $\Sigma_0^v = 5$, $\Sigma_0^{s_{i,0}} = 1$, $\Sigma_0^{u_n} = 3/N$, $\rho_0^{v,s_{i,0}} = 0.4$, $\rho_0^{s_{i,0},s_{j,0}} = 0$, and $\rho_0^{u_n,u_{n+1}} = 0.1$. The model is solved for four initial parametrizations: ($\rho_0^{v,u_n} = 0.2$, $\rho_0^{s_{i,0},u_n} = 0.1$), ($\rho_0^{v,u_n} = 0.1$, $\rho_0^{s_{i,0},u_n} = -0.1$), ($\rho_0^{v,u_n} = -0.1$, $\rho_0^{s_{i,0},u_n} = 0.1$), and ($\rho_0^{v,u_n} = -0.1$, $\rho_0^{s_{i,0},u_n} = -0.2$).

consistently informative about the asset value, prices reflect the most information at the end of the trading day. Moreover, uninformed orders stay conditionally highly autocorrelated and informed traders submit independent orders throughout most of the trading day. Finally, note that the conditional correlation between the asset value and uninformed orders is U-shaped and thus non-monotonic.

If uninformed orders are initially negatively correlated with the asset value but positively correlated with signals, the conditional correlations monotonically decrease over time. In that scenario, uninformed orders provide high camouflage for informed traders by pushing the price further away from the true asset value while losing their informational value about signals after the first few trading rounds. Also, the conditional uninformed order flow autocorrelation and the conditional signal correlation strongly decline over time.

Finally, suppose uninformed orders are initially negatively correlated with the asset value and signals. In that case, the conditional correlations stay approximately constant over time and only change towards the end of the trading day. Since $\rho_n^{v,u_{n+1}}$ and $\rho_n^{s_{i,0},u_{n+1}}$ remain persistently negative and only increase in the final periods, price discovery is low in the first trading rounds. Also, uninformed orders stay conditionally highly autocorrelated, and informed traders submit independent orders throughout most of the trading day.

1.5.3 Negative Initial Signal Correlation

In this analysis, the initial signal correlation equals $\rho_0^{s_{i,0},s_{j,0}} = -0.25$. For uninformed orders, the unconditional correlation structures are ($\rho_0^{v,u_n} = 0.4$, $\rho_0^{s_{i,0},u_n} = 0.15$), ($\rho_0^{v,u_n} = 0.2$, $\rho_0^{s_{i,0},u_n} = 0.1$), ($\rho_0^{v,u_n} = 0$, $\rho_0^{s_{i,0},u_n} = 0$), and ($\rho_0^{v,u_n} = -0.2$, $\rho_0^{s_{i,0},u_n} = -0.1$).[11]

Figure 1.5 plots informed traders' trading intensity β_n, the market maker's illiquidity parameter λ_n, the conditional asset variance Σ_n^v, and informed traders' terminal conditional expected profits over time. If uninformed orders are initially positively correlated with the asset value and signals, informed traders make the lowest expected profits and price discovery is the highest. Since aggregate order flows are most informative about the asset value in the first trading periods, the market maker's price sensitivity declines over time. It only increases towards the end of the trading day if the initial correlation between the asset value and uninformed orders is not too high. Interestingly, informed traders' trading intensity has a smirk shape if uninformed orders are initially highly correlated with the asset value. Therefore, trading intensities need not be monotonic.

[11] Previous numerical results included cases where ρ_0^{v,u_n} is positive while $\rho_0^{s_{i,0},u_n}$ is negative and vice versa. However, if signals are initially negatively correlated, the covariance matrix $\mathbf{\Sigma_0}$ in (1.1) is only positive definite given the specified parametrization if ρ_0^{v,u_n} and $\rho_0^{s_{i,0},u_n}$ have the same sign.

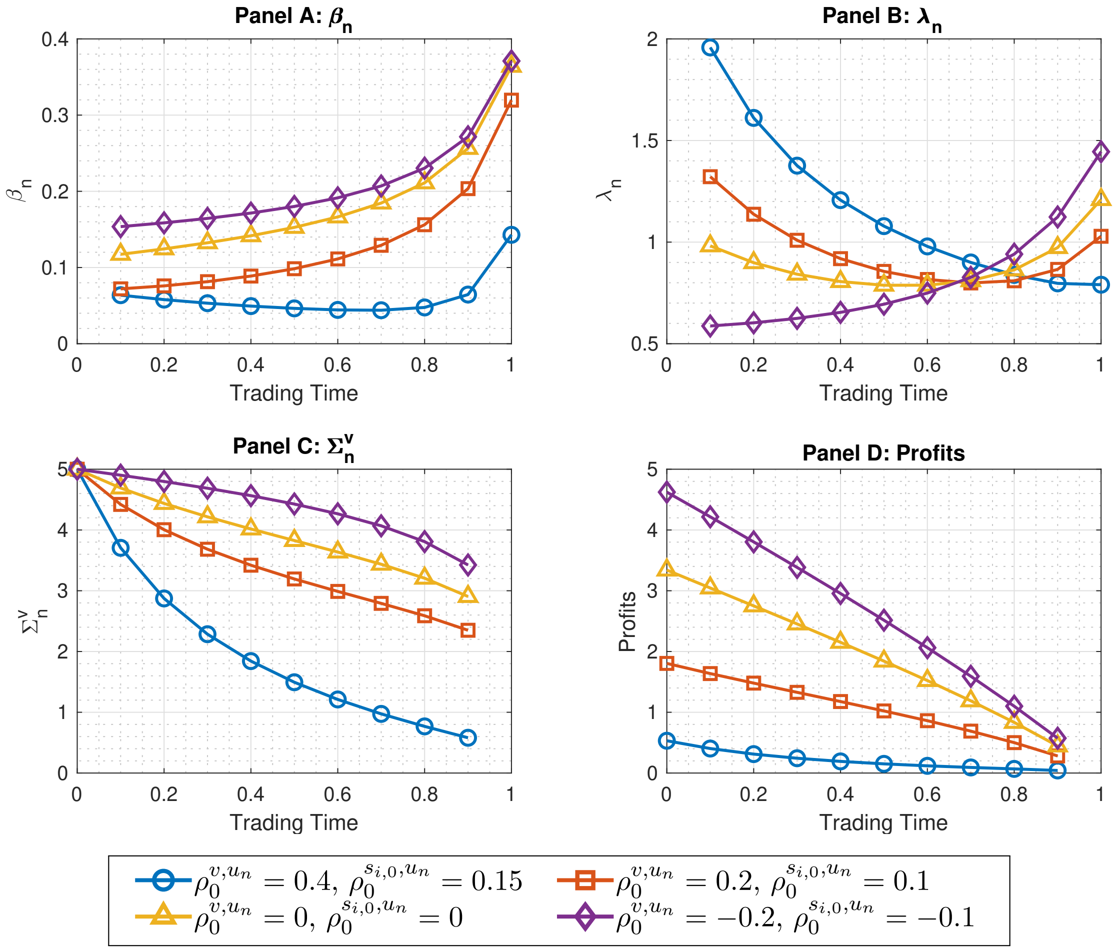

Figure 1.5: Trading Intensity, Illiquidity Parameter, Conditional Asset Variance, and Expected Profits with Negative Initial Signal Correlation. The figure plots informed traders' trading intensity β_n, the market maker's illiquidity parameter λ_n, the conditional asset variance Σ_n^v, and informed traders' terminal conditional expected profits over time. Parameter values are $M = 3$, $N = 10$, $\Sigma_0^v = 5$, $\Sigma_0^{s_{i,0}} = 1$, $\Sigma_0^{u_n} = 3/N$, $\rho_0^{v,s_{i,0}} = 0.4$, $\rho_0^{s_{i,0},s_{j,0}} = -0.25$, and $\rho_0^{u_n,u_{n+1}} = 0.1$. The model is solved for four initial parametrizations: $(\rho_0^{v,u_n} = 0.4,\ \rho_0^{s_{i,0},u_n} = 0.15)$, $(\rho_0^{v,u_n} = 0.2,\ \rho_0^{s_{i,0},u_n} = 0.1)$, $(\rho_0^{v,u_n} = 0,\ \rho_0^{s_{i,0},u_n} = 0)$, and $(\rho_0^{v,u_n} = -0.2,\ \rho_0^{s_{i,0},u_n} = -0.1)$.

If uninformed orders are initially uncorrelated with the asset value and signals, informed traders trade aggressively on their private information and make high expected profits. In that scenario, uninformed orders become conditionally negatively correlated with the asset value and signals over time. Thus, uninformed traders effectively conceal informed traders' orders by pushing the price away from the true asset value. Furthermore, the market maker's price sensitivity exhibits a clear U-shape. It initially declines because aggregate order flows are most informative about the asset value in the first periods. However, the illiquidity parameter eventually increases in later trading rounds since informed traders submit their most aggressive orders towards the end of the trading day.

Finally, if uninformed orders are initially negatively correlated with the asset value and signals, informed traders trade most aggressively on their residual signals and make the highest expected profits. Moreover, the market maker's illiquidity parameter monotonically increases over time, and price discovery is lowest at the end of the trading day. In that scenario, uninformed traders trade in the opposite direction of the true asset value and signals. As a result, informed traders can effectively conceal their orders from the market maker, who extracts the least information about the asset value from aggregate order flows.

Figure 1.6 shows the evolution of the conditional correlations $\rho_n^{u_{n+1},u_{n+2}}$, $\rho_n^{s_{i,0},s_{j,0}}$, $\rho_n^{v,u_{n+1}}$, and $\rho_n^{s_{i,0},u_{n+1}}$. If uninformed orders are initially positively correlated with the asset value and signals, $\rho_n^{u_{n+1},u_{n+2}}$, $\rho_n^{s_{i,0},s_{j,0}}$, and $\rho_n^{s_{i,0},u_{n+1}}$ monotonically decline over time. Interestingly, $\rho_n^{v,u_{n+1}}$ grows larger than its initial value in later periods. Consequently, uninformed orders can become even more informative about the asset value throughout the trading day.

Suppose uninformed orders are initially uncorrelated with the asset value and signals. In that case, the conditional correlations monotonically decrease over time. Specifically, uninformed orders become conditionally negatively correlated with the asset value and signals, providing camouflage for informed traders. Therefore, even if autocorrelated uninformed orders initially contain no information about the asset value and signals, they do so upon conditioning.

Finally, if uninformed orders are initially negatively correlated with the asset value and signals, the conditional signal correlation slowly declines, while the remaining conditional correlations stay approximately constant over time. It is interesting to note that the conditional uninformed order flow autocorrelation stays persistently high. Moreover, uninformed orders remain conditionally negatively correlated with the asset value and signals. In that scenario, informed traders can most effectively conceal their private information from the market maker and make the highest expected profits.

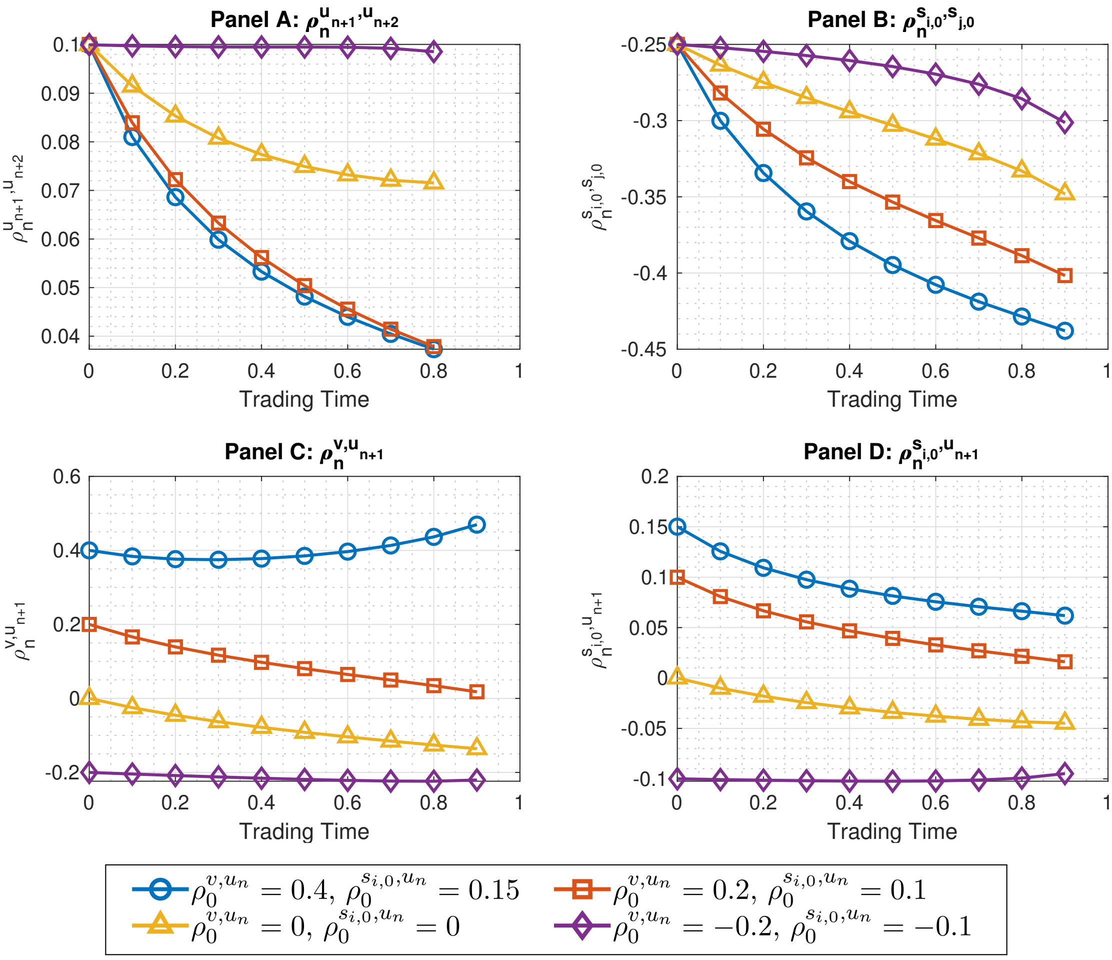

Figure 1.6: Conditional Correlations with Negative Initial Signal Correlation. The figure plots the conditional correlation between future uninformed orders $\rho_n^{u_{n+1},u_{n+2}}$, the conditional correlation between informed traders' signals $\rho_n^{s_{i,0},s_{j,0}}$, the conditional correlation between the asset value and future uninformed orders $\rho_n^{v,u_{n+1}}$, and the conditional correlation between informed traders' signals and future uninformed orders $\rho_n^{s_{i,0},u_{n+1}}$ over time. Parameter values are $M = 3$, $N = 10$, $\Sigma_0^v = 5$, $\Sigma_0^{s_{i,0}} = 1$, $\Sigma_0^{u_n} = 3/N$, $\rho_0^{v,s_{i,0}} = 0.4$, $\rho_0^{s_{i,0},s_{j,0}} = -0.25$, and $\rho_0^{u_n,u_{n+1}} = 0.1$. The model is solved for four initial parametrizations: ($\rho_0^{v,u_n} = 0.4$, $\rho_0^{s_{i,0},u_n} = 0.15$), ($\rho_0^{v,u_n} = 0.2$, $\rho_0^{s_{i,0},u_n} = 0.1$), ($\rho_0^{v,u_n} = 0$, $\rho_0^{s_{i,0},u_n} = 0$), and ($\rho_0^{v,u_n} = -0.2$, $\rho_0^{s_{i,0},u_n} = -0.1$).

1.6 Conclusion

This paper has derived a dynamic trading equilibrium with heterogeneously informed traders and (auto)correlated uninformed orders. A numerical analysis has shown that (auto)correlated uninformed orders significantly affect informed traders' trading intensities, the shape of the market maker's illiquidity parameter, and information efficiency in financial markets. In particular, autocorrelation in the uninformed order flow implies that uninformed orders are conditionally correlated with the asset value and informed traders' signals in the cross-section.

Since informed traders have an intrinsic motive to submit unpredictable orders, the empirically observed autocorrelation in order flows is likely to come from uninformed traders. Given the non-trivial effects of (auto)correlated uninformed orders on the dynamic trading equilibrium, this paper wants to emphasize that strategic trading models should not assume uninformed traders to be simply noise.

This paper's results provide testable implications for the dynamics of prices, order flows, and market depth. Moreover, this paper's model serves as a framework for interesting future research. Since this paper has studied a discrete-time framework, one natural extension is to solve the model in continuous time. Another extension is to relax the assumption of risk-neutral investors and allow for a more general utility function. Finally, the model only accounts for market orders, and allowing traders to submit limit orders may provide new valuable insights.

1.A Appendix

1.A.1 Proof of Lemma 1.1

Proof. Assume informed orders are as in (1.6). Then for all periods $n \in \{1, \dots, N\}$, one has

$$\begin{aligned} p_n &= \mathbb{E}\left[v \middle| y_{1:n}\right] \\ &= p_{n-1} + \mathbb{E}\left[v - p_{n-1} \middle| y_{1:n}\right] \\ &= p_{n-1} + \mathbb{E}\left[v - p_{n-1} \middle| y_{1:n-1}, y_n - r_{n,n-1}\right] \\ &= p_{n-1} + \mathbb{E}\left[v - p_{n-1} \middle| y_n - r_{n,n-1}\right] \\ &= p_{n-1} + \lambda_n (y_n - r_{n,n-1}). \end{aligned} \tag{1.A1}$$

In deriving this expression, the second line follows by definition since p_{n-1} is in the market maker's information set after n rounds of trading. The third line uses the fact that the information sets $y_{1:n}$ and $(y_{1:n-1}, y_n - r_{n,n-1})$ are equivalent since $r_{n,n-1}$ is a function of $y_{1:n-1}$. Going from the third to the fourth line, a direct application of the projection theorem for Gaussian random variables implies that both $v - p_{n-1} = v - \mathbb{E}\left[v \middle| y_{1:n-1}\right]$ and $y_n - r_{n,n-1} = y_n - \mathbb{E}\left[y_n \middle| y_{1:n-1}\right]$ are independent of $y_{1:n-1}$. Finally, λ_n is the regression coefficient from the projection of $v - p_{n-1}$ on $y_n - r_{n,n-1}$.

Similarly, for all $i \in \{1, \dots, M\}$ and all $n \in \{1, \dots, N\}$, one has

$$\begin{aligned} t_{i,n} &= \mathbb{E}\left[s_{i,0} \middle| y_{1:n}\right] \\ &= t_{i,n-1} + \mathbb{E}\left[s_{i,0} - t_{i,n-1} \middle| y_{1:n-1}, y_n - r_{n,n-1}\right] \\ &= t_{i,n-1} + \mathbb{E}\left[s_{i,n-1} \middle| y_n - r_{n,n-1}\right] \\ &= t_{i,n-1} + \zeta_n (y_n - r_{n,n-1}), \end{aligned} \tag{1.A2}$$

where ζ_n is the regression coefficient from the projection of $s_{i,n-1}$ on $y_n - r_{n,n-1}$.

Finally, for all $n \in \{1, \ldots, N-1\}$ and $t \in \{1, \ldots, N-n\}$, one has

$$\begin{aligned} r_{n+t,n} &= \mathbb{E}\left[u_{n+t} \middle| y_{1:n}\right] \\ &= r_{n+t,n-1} + \mathbb{E}\left[u_{n+t} - r_{n+t,n-1} \middle| y_{1:n-1}, y_n - r_{n,n-1}\right] \\ &= r_{n+t,n-1} + \mathbb{E}\left[u_{n+t} - r_{n+t,n-1} \middle| y_n - r_{n,n-1}\right] \\ &= r_{n,n-1} + \mathbb{E}\left[u_{n+1} - r_{n,n-1} \middle| y_n - r_{n,n-1}\right] \\ &= r_{n,n-1} + \theta_n(y_n - r_{n,n-1}), \end{aligned} \tag{1.A3}$$

where θ_n is the regression coefficient from the projection of $u_{n+1} - r_{n,n-1}$ on $y_n - r_{n,n-1}$. In going from the third to the fourth line, the distribution assumption (1.1) is used, which implies that u_{n+t} and u_{n+1} have the same conditional distribution for all $t \in \{1, \ldots, N-n\}$. Consequently, one also has $r_{n+t,n-1} = r_{n,n-1}$. □

1.A.2 Proof of Lemma 1.2

Proof. Define for all $n \in \{1, \ldots, N\}$ the following conditional (co)variances:

$$\Sigma_{n-1}^{y_n} \equiv \mathrm{Var}\left[y_n \middle| y_{1:n-1}\right], \tag{1.A4}$$

$$\Sigma_{n-1}^{s_{i,0},y_n} \equiv \mathrm{Cov}\left[s_{i,0}, y_n \middle| y_{1:n-1}\right]. \tag{1.A5}$$

Moreover, let $\Sigma_{n-1}^{\boldsymbol{s},y_n}$ be the $(M \times 1)$ vector with entries $\Sigma_{n-1}^{s_{i,0},y_n}$. Then the property of multivariate Gaussian distributions implies that the update of the $(M \times M)$ covariance matrix $\Sigma_n^{\boldsymbol{s}}$ with diagonals $\Sigma_n^{s_{i,0}}$ and off-diagonals $\Sigma_n^{s_{i,0},s_{j,0}}$ can be expressed recursively as

$$\Sigma_n^{\boldsymbol{s}} = \Sigma_{n-1}^{\boldsymbol{s}} - \Sigma_{n-1}^{\boldsymbol{s},y_n}\left(\Sigma_{n-1}^{y_n}\right)^{-1}\left(\Sigma_{n-1}^{\boldsymbol{s},y_n}\right)'. \tag{1.A6}$$

Since the entries in $\Sigma_{n-1}^{\boldsymbol{s},y_n}$ are all identical, it can be concluded that

$$\Sigma_{n-1}^{s_{i,0}} - \Sigma_n^{s_{i,0}} = \Sigma_{n-1}^{s_{i,0},s_{j,0}} - \Sigma_n^{s_{i,0},s_{j,0}}. \tag{1.A7}$$

A similar analysis yields

$$\Sigma_{n-1}^{u_n} - \Sigma_n^{u_{n+1}} = \Sigma_{n-1}^{u_n,u_{n+1}} - \Sigma_n^{u_{n+1},u_{n+2}}. \tag{1.A8}$$

Finally, to show the relation for λ_n, note that according to (1.8):

$$\begin{aligned}
&\lambda_n(y_n - r_{n,n-1})\\
=&p_n - p_{n-1}\\
=&\mathbb{E}\left[v - p_{n-1}\middle|y_{1:n}\right]\\
=&\mathbb{E}\left[\mathbb{E}\left[v - p_{n-1}\middle|s_{1,0},\dots,s_{M,0},u_n,y_{1:n-1}\right]\middle|y_{1:n}\right]\\
=&\mathbb{E}\left[\mathbb{E}\left[v - p_{n-1}\middle|s_{1,n-1},\dots,s_{M,n-1},u_n - r_{n,n-1},y_{1:n-1}\right]\middle|y_{1:n-1},y_n - r_{n,n-1}\right]\\
=&\mathbb{E}\left[\mathbb{E}\left[v - p_{n-1}\middle|s_{1,n-1},\dots,s_{M,n-1},u_n - r_{n,n-1}\right]\middle|y_n - r_{n,n-1}\right]\\
=&\mathbb{E}\left[\begin{pmatrix}\Sigma_{n-1}^{v,s} & \Sigma_{n-1}^{v,u_n}\end{pmatrix}\begin{pmatrix}\Sigma_{n-1}^{s} & \Sigma_{n-1}^{s,u_n}\\ \Sigma_{n-1}^{u_n,s} & \Sigma_{n-1}^{u_n}\end{pmatrix}^{-1}\begin{pmatrix}s_{1,n-1}\\ \vdots\\ s_{M,n-1}\\ u_n - r_{n,n-1}\end{pmatrix}\middle|y_n - r_{n,n-1}\right]\\
=&\begin{pmatrix}\Sigma_{n-1}^{v,s} & \Sigma_{n-1}^{v,u_n}\end{pmatrix}\begin{pmatrix}\Sigma_{n-1}^{s} & \Sigma_{n-1}^{s,u_n}\\ \Sigma_{n-1}^{u_n,s} & \Sigma_{n-1}^{u_n}\end{pmatrix}^{-1}\begin{pmatrix}\zeta_n\\ \vdots\\ \zeta_n\\ \tilde{\theta}_n\end{pmatrix}(y_n - r_{n,n-1}).
\end{aligned} \tag{1.A9}$$

In deriving the above expression, the fourth line follows from the law of iterated expectations since the information set $y_{1:n}$ is a subset of $(s_{1,0},\dots,s_{M,0},u_n,y_{1:n-1})$. Also, these information sets can be equivalently expressed as $(y_{1:n-1},y_n - r_{n,n-1})$ and $(s_{1,n-1},\dots,s_{M,n-1},u_n - r_{n,n-1},y_{1:n-1})$, respectively. Then the projection theorem for Gaussian random variables implies that $v - p_{n-1}$, $(s_{1,n-1},\dots,s_{M,n-1},u_n - r_{n,n-1})$, and $y_n - r_{n,n-1}$ are all independent of $y_{1:n-1}$. Finally, the last two lines follow from the property of multivariate Gaussian distributions, and $\tilde{\theta}_n$ is the regression coefficient from the projection of $u_n - r_{n,n-1}$ on $y_n - r_{n,n-1}$. □

1.A.3 Proof of Lemma 1.3

Proof. The proof is analogous to Foster and Viswanathan (1996). In every period $n-1$ where $n \in \{1,\dots,N\}$ every informed trader $i \in \{1,\dots,M\}$ forecasts the next period's uninformed order flow u_n as follows:

$$\begin{aligned}
\mathbb{E}\left[u_n \middle| s_{i,0}, y_{1:n-1}, x_{i,1:n-1}\right] &= \mathbb{E}\left[u_n \middle| s_{i,0}, y_{1:n-1}\right] \\
&= r_{n,n-1} + \mathbb{E}\left[u_n - r_{n,n-1} \middle| s_{i,0}, y_{1:n-1}\right] \\
&= r_{n,n-1} + \mathbb{E}\left[u_n - r_{n,n-1} \middle| s_{i,n-1}, y_{1:n-1}\right] \\
&= r_{n,n-1} + \mathbb{E}\left[u_n - r_{n,n-1} \middle| s_{i,n-1}\right] \\
&= r_{n,n-1} + \psi_n s_{i,n-1}.
\end{aligned} \tag{1.A10}$$

The first line exploits the result that the trading history $x_{i,1:n-1}$ is redundant information, as was shown in (1.17). The second line follows by definition since $r_{n,n-1}$ is in informed trader i's information set after $n-1$ periods of trading. The third line uses the fact that the information sets $(s_{i,0}, y_{1:n-1})$ and $(s_{i,n-1}, y_{1:n-1})$ are equivalent since $s_{i,n-1} = s_{i,0} - t_{i,n-1}$ and $t_{i,n-1}$ is a function of $y_{1:n-1}$. Finally, the last two lines follow from the projection theorem for Gaussian random variables and ψ_n is the regression coefficient from the projection of $u_n - r_{n,n-1}$ on $s_{i,n-1}$.

Similarly, informed trader i's update of the asset value v equals

$$\begin{aligned}
\mathbb{E}\left[v \middle| s_{i,0}, y_{1:n-1}, x_{i,1:n-1}\right] &= p_{n-1} + \mathbb{E}\left[v - p_{n-1} \middle| s_{i,n-1}\right] \\
&= p_{n-1} + \eta_n s_{i,n-1},
\end{aligned} \tag{1.A11}$$

where η_n is the regression coefficient from the projection of $v - p_{n-1}$ on $s_{i,n-1}$.

Moreover, informed trader i's belief about informed trader j's signal $s_{j,0}$, where $j \in \{1,\dots,M\}$ and $j \neq i$, equals

$$\begin{aligned}
\mathbb{E}\left[s_{j,0} \middle| s_{i,0}, y_{1:n-1}, x_{i,1:n-1}\right] &= t_{j,n-1} + \mathbb{E}\left[s_{j,n-1} \middle| s_{i,n-1}\right] \\
&= t_{j,n-1} + \phi_n s_{i,n-1},
\end{aligned} \tag{1.A12}$$

where ϕ_n is the regression coefficient from the projection of $s_{j,n-1}$ on $s_{i,n-1}$. □

1.A.4 Proof of Lemma 1.4

Proof. The proof is taken from Foster and Viswanathan (1996) and is shown by induction. Consider the aggregate order flow in the first trading period as the base case:

$$y_1 = \sum_{j\neq i} \beta_1 s_{j,0} + \tilde{x}_{i,1} + u_1 = \sum_{j\neq i} \beta_1 \hat{s}^i_{j,0} + \beta_1 \hat{s}^i_{i,0} + (\tilde{x}_{i,1} - \beta_1 \hat{s}^i_{i,0}) + u_1 = \hat{y}^i_1 + (\tilde{x}_{i,1} - \beta_1 s_{i,0}). \tag{1.A13}$$

Since $(s_{i,0}, y_1, \tilde{x}_{i,1})$ is in informed trader i's information set after the first trading period, so is $\hat{y}^i_1$.

Assume as induction hypothesis that $\hat{y}^i_{1:n-1}$ is in informed trader i's information set after $n-1$ periods where $n \in \{2, \ldots, N\}$. To show that $\hat{y}^i_n$ is in informed trader i's information set after n trading periods, write the aggregate order flow in period n as

$$\begin{aligned}
y_n &= \sum_{j\neq i} \beta_n s_{j,n-1} + \tilde{x}_{i,n} + u_n \\
&= \sum_{j\neq i} \beta_n (s_{j,0} - t_{j,n-1}) + \tilde{x}_{i,n} + u_n \\
&= \sum_{j\neq i} \beta_n \left(s_{j,0} - \hat{t}^i_{j,n-1}\right) + \sum_{j\neq i} \beta_n \left(\hat{t}^i_{j,n-1} - t_{j,n-1}\right) + \tilde{x}_{i,n} + u_n \\
&= \sum_{j\neq i} \beta_n \hat{s}^i_{j,n-1} + \beta_n \hat{s}^i_{i,n-1} + u_n + \sum_{j\neq i} \beta_n \left(\hat{t}^i_{j,n-1} - t_{j,n-1}\right) + \tilde{x}_{i,n} - \beta_n \hat{s}^i_{i,n-1} \\
&= \hat{y}^i_n + \sum_{j\neq i} \beta_n \left(\hat{t}^i_{j,n-1} - t_{j,n-1}\right) + \tilde{x}_{i,n} - \beta_n \left(s_{i,0} - \hat{t}^i_{i,n-1}\right).
\end{aligned} \tag{1.A14}$$

Note that $\hat{t}^i_{j,n-1}$ is a function of the equilibrium order flow history $\hat{y}^i_{1:n-1}$, which is in informed trader i's information set by the induction hypothesis. Since informed trader i knows $(s_{i,0}, y_{1:n}, \tilde{x}_{i,1:n})$ and thus also $t_{j,n-1}$ after n trading periods, it can be concluded that $\hat{y}^i_n$ is also in informed trader i's information set. □

1.A.5 Proof of Lemma 1.5

Proof. The proof is analogous to Foster and Viswanathan (1996). Suppose informed trader i has submitted an arbitrary order sequence $\tilde{x}_{i,1:n-1}$ in the first $n-1$ periods where $n \in \{2, \ldots, N\}$. Then

$$
\begin{aligned}
&\mathbb{E}\left[u_n - r_{n,n-1} \middle| s_{i,0}, y_{1:n-1}, \tilde{x}_{i,1:n-1}\right] \\
=&\mathbb{E}\left[u_n - \hat{r}^i_{n,n-1} + \hat{r}^i_{n,n-1} - r_{n,n-1} \middle| s_{i,0}, y_{1:n-1}, \tilde{x}_{i,1:n-1}\right] \\
=&\mathbb{E}\left[u_n - \hat{r}^i_{n,n-1} \middle| s_{i,0}, y_{1:n-1}, \tilde{x}_{i,1:n-1}\right] + \left(\hat{r}^i_{n,n-1} - r_{n,n-1}\right).
\end{aligned}
\tag{1.A15}
$$

Note that Lemma 1.4 has shown that $\hat{y}^i_{1:n-1}$ can be recursively constructed from the information set $(s_{i,0}, y_{1:n-1}, \tilde{x}_{i,1:n-1})$, which implies that $\hat{r}^i_{n,n-1} - r_{n,n-1}$ is in informed trader i's information set after $n-1$ periods. Moreover, the term on the left-hand side equals

$$
\begin{aligned}
&\mathbb{E}\left[u_n - \hat{r}^i_{n,n-1} \middle| s_{i,0}, y_{1:n-1}, \tilde{x}_{i,1:n-1}\right] \\
=&\mathbb{E}\left[u_n - \hat{r}^i_{n,n-1} \middle| s_{i,0}, \hat{y}^i_{1:n-1}, \tilde{x}_{i,1:n-1}\right] \\
=&\mathbb{E}\left[u_n - \hat{r}^i_{n,n-1} \middle| \hat{s}^i_{i,n-1}, \hat{y}^i_{1:n-1}, \tilde{x}_{i,1:n-1}\right] \\
=&\mathbb{E}\left[u_n - \hat{r}^i_{n,n-1} \middle| \hat{s}^i_{i,n-1}, \tilde{x}_{i,1:n-1}\right] \\
=&\mathbb{E}\left[u_n - \hat{r}^i_{n,n-1} \middle| \hat{s}^i_{i,n-1}\right] \\
=&\psi_n \hat{s}^i_{i,n-1},
\end{aligned}
\tag{1.A16}
$$

where the penultimate line makes use of the fact that $\tilde{x}_{i,1:n-1}$ is an arbitrary sequence of numbers that is unrelated to equilibrium play. Therefore, it can be concluded that

$$
\mathbb{E}\left[u_n - r_{n,n-1} \middle| s_{i,0}, y_{1:n-1}, \tilde{x}_{i,1:n-1}\right] = \psi_n \hat{s}^i_{i,n-1} + \left(\hat{r}^i_{n,n-1} - r_{n,n-1}\right). \tag{1.A17}
$$

A similar analysis shows that

$$
\mathbb{E}\left[v - p_{n-1} \middle| s_{i,0}, y_{1:n-1}, \tilde{x}_{i,1:n-1}\right] = \eta_n \hat{s}^i_{i,n-1} + \left(\hat{p}^i_{n-1} - p_{n-1}\right), \tag{1.A18}
$$

and

$$\mathbb{E}\left[s_{j,n-1}\middle|s_{i,0}, y_{1:n-1}, \tilde{x}_{i,1:n-1}\right] = \phi_n \hat{s}^i_{i,n-1} + \left(\hat{t}^i_{j,n-1} - t_{j,n-1}\right), \tag{1.A19}$$

as was to be shown. □

1.A.6 Proof of Proposition 1.1

Proof. Given the conjectured value function (1.26), informed trader i's Bellman equation in trading period $n \in \{1, \ldots, N\}$ writes

$$\begin{aligned}
&\max_{x_{i,n}} \mathbb{E}\left[(v - p_n)x_{i,n}\middle|s_{i,0}, y_{1:n-1}, \tilde{x}_{i,1:n-1}\right] \\
&+ \mathbb{E}\Big[a_n\left(\hat{s}^i_{i,n}\right)^2 + b_n\left(\hat{p}^i_n - p_n\right)^2 + c_n\left(\hat{t}^i_{j,n} - t_{j,n}\right)^2 + d_n\left(\hat{r}^i_{n+1,n} - r_{n+1,n}\right)^2 \\
&\quad + e_n\hat{s}^i_{i,n}\left(\hat{p}^i_n - p_n\right) + f_n\hat{s}^i_{i,n}\left(\hat{t}^i_{j,n} - t_{j,n}\right) + g_n\hat{s}^i_{i,n}\left(\hat{r}^i_{n+1,n} - r_{n+1,n}\right) \\
&\quad + h_n\left(\hat{p}^i_n - p_n\right)\left(\hat{t}^i_{j,n} - t_{j,n}\right) + i_n\left(\hat{p}^i_n - p_n\right)\left(\hat{r}^i_{n+1,n} - r_{n+1,n}\right) \\
&\quad + j_n\left(\hat{t}^i_{j,n} - t_{j,n}\right)\left(\hat{r}^i_{n+1,n} - r_{n+1,n}\right) + k_n\Big|s_{i,0}, y_{1:n-1}, \tilde{x}_{i,1:n-1}\Big].
\end{aligned} \tag{1.A20}$$

Under the conjectured equilibrium, one has

$$p_n = p_{n-1} + \lambda_n(y_n - r_{n,n-1}) = p_{n-1} + \lambda_n\left(x_{i,n} + \sum_{j\neq i}\beta_n s_{j,n-1} + u_n - r_{n,n-1}\right), \tag{1.A21}$$

and

$$t_{j,n} = t_{j,n-1} + \zeta_n(y_n - r_{n,n-1}) = t_{j,n-1} + \zeta_n\left(x_{i,n} + \sum_{j\neq i}\beta_n s_{j,n-1} + u_n - r_{n,n-1}\right), \tag{1.A22}$$

and[12]

$$r_{n+1,n} = r_{n,n-1} + \theta_n(y_n - r_{n,n-1}) = r_{n,n-1} + \theta_n\left(x_{i,n} + \sum_{j\neq i}\beta_n s_{j,n-1} + u_n - r_{n,n-1}\right). \tag{1.A23}$$

[12]By definition, $r_{N+1,N} = 0$ since there are no uninformed orders after the final trading period.

Taking the first derivative with respect to $x_{i,n}$ yields the first-order condition

$$
\begin{aligned}
&\mathbb{E}\left[v - p_{n-1} - \lambda_n\left(\sum_{j\neq i}\beta_n s_{j,n-1} + u_n - r_{n,n-1}\right)\middle| s_{i,0}, y_{1:n-1}, \tilde{x}_{i,1:n-1}\right] - 2\lambda_n x_{i,n} \\
&+\mathbb{E}\Big[2b_n\left(\hat{p}^i_n - p_n\right)(-\lambda_n) + 2c_n\left(\hat{t}^i_{j,n} - t_{j,n}\right)(-\zeta_n) + 2d_n\left(\hat{r}^i_{n+1,n} - r_{n+1,n}\right)(-\theta_n) \\
&\quad + e_n\hat{s}^i_{i,n}(-\lambda_n) + f_n\hat{s}^i_{i,n}(-\zeta_n) + g_n\hat{s}^i_{i,n}(-\theta_n) + h_n\left[-\lambda_n\left(\hat{t}^i_{j,n} - t_{j,n}\right) - \zeta_n\left(\hat{p}^i_n - p_n\right)\right] \\
&\quad + i_n\left[-\lambda_n\left(\hat{r}^i_{n+1,n} - r_{n+1,n}\right) - \theta_n\left(\hat{p}^i_n - p_n\right)\right] + j_n\Big[-\zeta_n\left(\hat{r}^i_{n+1,n} - r_{n+1,n}\right) \\
&\quad -\theta_n\left(\hat{t}^i_{j,n} - t_{j,n}\right)\Big]\Big| s_{i,0}, y_{1:n-1}, \tilde{x}_{i,1:n-1}\Big] = 0.
\end{aligned}
\tag{1.A24}
$$

Similarly, one obtains the second-order condition

$$
-\lambda_n(2-2b_n\lambda_n-h_n\zeta_n-i_n\theta_n)+\zeta_n(2c_n\zeta_n+h_n\lambda_n+j_n\theta_n)+\theta_n(2d_n\theta_n+i_n\lambda_n+j_n\zeta_n) < 0.
\tag{1.A25}
$$

Following Foster and Viswanathan (1996), expand the first-order condition:

$$
\begin{aligned}
&\mathbb{E}\left[v - \hat{p}^i_{n-1} - \lambda_n\left(\sum_{j\neq i}\beta_n \hat{s}^i_{j,n-1} + u_n - \hat{r}^i_{n,n-1}\right)\middle| s_{i,0}, y_{1:n-1}, \tilde{x}_{i,1:n-1}\right] \\
&\quad - 2\lambda_n x_{i,n} + \left(\hat{p}^i_{n-1} - p_{n-1}\right) + \lambda_n\left[\sum_{j\neq i}\beta_n\left(\hat{s}^i_{j,n-1} - s_{j,n-1}\right) - \left(\hat{r}^i_{n,n-1} - r_{n,n-1}\right)\right] \\
&-\mathbb{E}\Big[(e_n\lambda_n + f_n\zeta_n + g_n\theta_n)\hat{s}^i_{i,n} + (2b_n\lambda_n + h_n\zeta_n + i_n\theta_n)\left(\hat{p}^i_n - p_n\right) \\
&\quad + (2c_n\zeta_n + h_n\lambda_n + j_n\theta_n)\left(\hat{t}^i_{j,n} - t_{j,n}\right) + (2d_n\theta_n + i_n\lambda_n + j_n\zeta_n)\left(\hat{r}^i_{n+1,n} - r_{n+1,n}\right) \\
&\quad \Big| s_{i,0}, y_{1:n-1}, \tilde{x}_{i,1:n-1}\Big] = 0,
\end{aligned}
\tag{1.A26}
$$

where $\hat{p}^i_{n-1} - p_{n-1}$, $\hat{s}^i_{j,n-1} - s_{j,n-1}$, and $\hat{r}^i_{n,n-1} - r_{n,n-1}$ are in informed trader i's information set. Before proceeding with the first-order condition, recall from Lemma 1.3 that

$$
\mathbb{E}\left[v - \hat{p}^i_{n-1}\middle| s_{i,0}, y_{1:n-1}, \tilde{x}_{i,1:n-1}\right] = \eta_n\hat{s}^i_{i,n-1}, \tag{1.A27}
$$

$$
\mathbb{E}\left[\hat{s}^i_{j,n-1}\middle| s_{i,0}, y_{1:n-1}, \tilde{x}_{i,1:n-1}\right] = \phi_n\hat{s}^i_{i,n-1}, \tag{1.A28}
$$

$$
\mathbb{E}\left[u_n - \hat{r}^i_{n,n-1}\middle| s_{i,0}, y_{1:n-1}, \tilde{x}_{i,1:n-1}\right] = \psi_n\hat{s}^i_{i,n-1}, \tag{1.A29}
$$

where η_n, ϕ_n, and ψ_n are the corresponding regression coefficients. With these results, one can show that

$$\begin{aligned}
&\mathbb{E}\left[\hat{s}^i_{i,n}\middle|s_{i,0}, y_{1:n-1}, \tilde{x}_{i,1:n-1}\right] \\
=&\hat{s}^i_{i,n-1} - \zeta_n\,\mathbb{E}\left[\beta_n\hat{s}^i_{i,n-1} + \sum_{j\neq i}\beta_n\hat{s}^i_{j,n-1} + u_n - \hat{r}^i_{n,n-1}\middle|\hat{s}^i_{i,n-1}\right] \\
=&\Big(1 - \zeta_n\Big[\beta_n + (M-1)\beta_n\phi_n + \psi_n\Big]\Big)\hat{s}^i_{i,n-1}.
\end{aligned} \tag{1.A30}$$

Moreover,

$$\begin{aligned}
&\hat{p}^i_n - p_n = \hat{p}^i_{n-1} - p_{n-1} + \lambda_n\left[\hat{y}^i_n - \hat{r}^i_{n,n-1} - (y_n - r_{n,n-1})\right] \\
=&\hat{p}^i_{n-1} - p_{n-1} + \lambda_n\Big[\beta_n\hat{s}^i_{i,n-1} - x_{i,n} - (M-1)\beta_n\left(\hat{t}^i_{j,n-1} - t_{j,n-1}\right) - \left(\hat{r}^i_{n,n-1} - r_{n,n-1}\right)\Big],
\end{aligned} \tag{1.A31}$$

and

$$\begin{aligned}
&\hat{t}^i_{j,n} - t_{j,n} = \hat{t}^i_{j,n-1} - t_{j,n-1} + \zeta_n\left[\hat{y}^i_n - \hat{r}^i_{n,n-1} - (y_n - r_{n,n-1})\right] \\
=&\hat{t}^i_{j,n-1} - t_{j,n-1} + \zeta_n\Big[\beta_n\hat{s}^i_{i,n-1} - x_{i,n} - (M-1)\beta_n\left(\hat{t}^i_{j,n-1} - t_{j,n-1}\right) - \left(\hat{r}^i_{n,n-1} - r_{n,n-1}\right)\Big],
\end{aligned} \tag{1.A32}$$

and

$$\begin{aligned}
&\hat{r}^i_{n+1,n} - r_{n+1,n} = \hat{r}^i_{n,n-1} - r_{n,n-1} + \theta_n\left[\hat{y}^i_n - \hat{r}^i_{n,n-1} - (y_n - r_{n,n-1})\right] \\
=&\hat{r}^i_{n,n-1} - r_{n,n-1} + \theta_n\Big[\beta_n\hat{s}^i_{i,n-1} - x_{i,n} - (M-1)\beta_n\left(\hat{t}^i_{j,n-1} - t_{j,n-1}\right) - \left(\hat{r}^i_{n,n-1} - r_{n,n-1}\right)\Big].
\end{aligned} \tag{1.A33}$$

Using all of these relations in the first-order condition, one can show that informed trader i's optimal strategy in period n equals

$$x_{i,n} = \beta_n\hat{s}^i_{i,n-1} + \gamma_n\left(\hat{p}^i_{n-1} - p_{n-1}\right) + \alpha_n\left(\hat{t}^i_{j,n-1} - t_{j,n-1}\right) + \delta_n\left(\hat{r}^i_{n,n-1} - r_{n,n-1}\right), \tag{1.A34}$$

where the coefficients β_n, γ_n, α_n, and δ_n are just as in Proposition 1.1. Finally, plugging $x_{i,n}$ into the Bellman equation and noting that

$$\begin{aligned}
&\mathbb{E}\left[\left(\hat{s}^i_{i,n}\right)^2\middle|s_{i,0},y_{1:n-1},\tilde{x}_{i,1:n-1}\right]\\
=&\Big(1-\zeta_n\Big[\beta_n+(M-1)\beta_n\phi_n+\psi_n\Big]\Big)^2\left(\hat{s}^i_{i,n-1}\right)^2\\
&+\zeta_n^2\Bigg[\operatorname{Var}\left[u_n-\hat{r}^i_{n,n-1}\middle|s_{i,0},\hat{y}^i_{1:n-1}\right]+2(M-1)\beta_n\operatorname{Cov}\left[\hat{s}^i_{j,n-1},u_n-\hat{r}^i_{n,n-1}\middle|s_{i,0},\hat{y}^i_{1:n-1}\right]\\
&+(M-1)\beta_n^2\Big(\operatorname{Var}\left[\hat{s}^i_{j,n-1}\middle|s_{i,0},\hat{y}^i_{1:n-1}\right]+(M-2)\operatorname{Cov}\left[\hat{s}^i_{j,n-1},\hat{s}^i_{k,n-1}\middle|s_{i,0},\hat{y}^i_{1:n-1}\right]\Big)\Bigg],
\end{aligned} \tag{1.A35}$$

one obtains the value function coefficients in period $n-1$. At this stage of the proof, the "hat" notation is suppressed because informed trader i will not deviate from his optimal strategy along the equilibrium path. The regression coefficients η_n, ϕ_n, and ψ_n are defined by projecting $v-p_{n-1}$, $s_{j,n-1}$, and $u_n-r_{n,n-1}$ on $s_{i,n-1}$, respectively. Consequently,

$$\eta_n=\frac{\operatorname{Cov}\left[v-p_{n-1},s_{i,n-1}\right]}{\operatorname{Var}\left[s_{i,n-1}\right]}=\frac{\Sigma^{v,s_{i,0}}_{n-1}}{\Sigma^{s_{i,0}}_{n-1}}, \tag{1.A36}$$

$$\phi_n=\frac{\operatorname{Cov}\left[s_{j,n-1},s_{i,n-1}\right]}{\operatorname{Var}\left[s_{i,n-1}\right]}=\frac{\Sigma^{s_{j,0},s_{i,0}}_{n-1}}{\Sigma^{s_{i,0}}_{n-1}}, \tag{1.A37}$$

$$\psi_n=\frac{\operatorname{Cov}\left[u_n-r_{n,n-1},s_{i,n-1}\right]}{\operatorname{Var}\left[s_{i,n-1}\right]}=\frac{\Sigma^{u_n,s_{i,0}}_{n-1}}{\Sigma^{s_{i,0}}_{n-1}}. \tag{1.A38}$$

Next, one obtains

$$\begin{aligned}
&\operatorname{Var}\left[u_n-r_{n,n-1}\middle|s_{i,0},y_{1:n-1}\right]\\
=&\,\mathbb{E}\left[(u_n-r_{n,n-1}-\psi_n s_{i,n-1})^2\right]\\
=&\,\mathbb{E}\left[(u_n-r_{n,n-1})^2\right]-2\psi_n\,\mathbb{E}\left[(u_n-r_{n,n-1})s_{i,n-1}\right]+\psi_n^2\,\mathbb{E}\left[(s_{i,n-1})^2\right]\\
=&\,\Sigma^{u_n}_{n-1}-2\psi_n\Sigma^{u_n,s_{i,0}}_{n-1}+\psi_n^2\Sigma^{s_{i,0}}_{n-1}\\
=&\,\Sigma^{u_n}_{n-1}-\psi_n^2\Sigma^{s_{i,0}}_{n-1},
\end{aligned} \tag{1.A39}$$

and

$$
\begin{aligned}
&\operatorname{Var}\left[s_{j,n-1}\middle|s_{i,0},y_{1:n-1}\right]\\
=&\,\mathbb{E}\left[(s_{j,n-1}-\phi_n s_{i,n-1})^2\right]\\
=&\,\mathbb{E}\left[(s_{j,n-1})^2\right]-2\phi_n\,\mathbb{E}\left[s_{j,n-1}s_{i,n-1}\right]+\phi_n^2\,\mathbb{E}\left[(s_{i,n-1})^2\right]\\
=&\Sigma_{n-1}^{s_{j,0}}-2\phi_n\Sigma_{n-1}^{s_{j,0},s_{i,0}}+\phi_n^2\Sigma_{n-1}^{s_{i,0}}\\
=&(1-\phi_n^2)\Sigma_{n-1}^{s_{i,0}},
\end{aligned}
\tag{1.A40}
$$

and

$$
\begin{aligned}
&\operatorname{Cov}\left[s_{j,n-1},s_{k,n-1}\middle|s_{i,0},y_{1:n-1}\right]\\
=&\operatorname{Cov}\left[s_{j,n-1}-\phi_n s_{i,n-1},s_{k,n-1}-\phi_n s_{i,n-1}\right]\\
=&\operatorname{Cov}\left[s_{j,n-1},s_{i,n-1}\right]-2\phi_n\operatorname{Cov}\left[s_{j,n-1},s_{i,n-1}\right]+\phi_n^2\operatorname{Cov}\left[s_{i,n-1},s_{i,n-1}\right]\\
=&\Sigma_{n-1}^{s_{j,0},s_{i,0}}-2\phi_n\Sigma_{n-1}^{s_{j,0},s_{i,0}}+\phi_n^2\Sigma_{n-1}^{s_{i,0}}\\
=&(1-\phi_n)\Sigma_{n-1}^{s_{j,0},s_{i,0}},
\end{aligned}
\tag{1.A41}
$$

and

$$
\begin{aligned}
&\operatorname{Cov}\left[s_{j,n-1},u_n-r_{n,n-1}\middle|s_{i,0},y_{1:n-1}\right]\\
=&\operatorname{Cov}\left[s_{j,n-1}-\phi_n s_{i,n-1},u_n-r_{n,n-1}-\psi_n s_{i,n-1}\right]\\
=&\operatorname{Cov}\left[s_{j,n-1},u_n-r_{n,n-1}\right]-\phi_n\operatorname{Cov}\left[s_{i,n-1},u_n-r_{n,n-1}\right]\\
&-\psi_n\operatorname{Cov}\left[s_{j,n-1},s_{i,n-1}\right]+\phi_n\psi_n\operatorname{Cov}\left[s_{i,n-1},s_{i,n-1}\right]\\
=&\Sigma_{n-1}^{s_{i,0},u_n}-\phi_n\Sigma_{n-1}^{s_{i,0},u_n}-\psi_n\Sigma_{n-1}^{s_{j,0},s_{i,0}}+\phi_n\psi_n\Sigma_{n-1}^{s_{i,0}}\\
=&(1-\phi_n)\Sigma_{n-1}^{s_{i,0},u_n}.
\end{aligned}
\tag{1.A42}
$$

To compute λ_n, recall that

$$
y_n-r_{n,n-1}=\sum_{i=1}^{M}\beta_n s_{i,n-1}+u_n-r_{n,n-1}. \tag{1.A43}
$$

Then the projection theorem implies

$$\lambda_n = \frac{\text{Cov}\left[v - p_{n-1}, y_n - r_{n,n-1}\right]}{\text{Var}\left[y_n - r_{n,n-1}\right]} = \frac{M\beta_n \Sigma_{n-1}^{v,s_{i,0}} + \Sigma_{n-1}^{v,u_n}}{M\beta_n^2\left[\Sigma_{n-1}^{s_{i,0}} + (M-1)\Sigma_{n-1}^{s_{i,0},s_{j,0}}\right] + \Sigma_{n-1}^{u_n} + 2M\beta_n\Sigma_{n-1}^{s_{i,0},u_n}}. \tag{1.A44}$$

Similarly,

$$\zeta_n = \frac{\text{Cov}\left[s_{i,n-1}, y_n - r_{n,n-1}\right]}{\text{Var}\left[y_n - r_{n,n-1}\right]} = \frac{\beta_n\left[\Sigma_{n-1}^{s_{i,0}} + (M-1)\Sigma_{n-1}^{s_{i,0},s_{j,0}}\right] + \Sigma_{n-1}^{s_{i,0},u_n}}{M\beta_n^2\left[\Sigma_{n-1}^{s_{i,0}} + (M-1)\Sigma_{n-1}^{s_{i,0},s_{j,0}}\right] + \Sigma_{n-1}^{u_n} + 2M\beta_n\Sigma_{n-1}^{s_{i,0},u_n}}, \tag{1.A45}$$

and[13]

$$\theta_n = \frac{\text{Cov}\left[u_{n+1} - r_{n,n-1}, y_n - r_{n,n-1}\right]}{\text{Var}\left[y_n - r_{n,n-1}\right]} = \frac{M\beta_n \Sigma_{n-1}^{s_{i,0},u_n} + \Sigma_{n-1}^{u_n,u_{n+1}}}{M\beta_n^2\left[\Sigma_{n-1}^{s_{i,0}} + (M-1)\Sigma_{n-1}^{s_{i,0},s_{j,0}}\right] + \Sigma_{n-1}^{u_n} + 2M\beta_n\Sigma_{n-1}^{s_{i,0},u_n}}. \tag{1.A46}$$

Finally, to compute the updates of conditional (co)variances, define for all periods $n \in \{1, \dots, N-1\}$:

$$\Sigma_{n-1}^{y_n} \equiv \text{Var}\left[y_n \middle| y_{1:n-1}\right] = \text{Var}\left[y_n - r_{n,n-1}\right], \tag{1.A47}$$

$$\Sigma_{n-1}^{v,y_n} \equiv \text{Cov}\left[v, y_n \middle| y_{1:n-1}\right] = \text{Cov}\left[v - p_{n-1}, y_n - r_{n,n-1}\right], \tag{1.A48}$$

$$\Sigma_{n-1}^{s_{i,0},y_n} \equiv \text{Cov}\left[s_{i,0}, y_n \middle| y_{1:n-1}\right] = \text{Cov}\left[s_{i,n-1}, y_n - r_{n,n-1}\right], \tag{1.A49}$$

$$\Sigma_{n-1}^{u_{n+1},y_n} \equiv \text{Cov}\left[u_{n+1}, y_n \middle| y_{1:n-1}\right] = \text{Cov}\left[u_{n+1} - r_{n,n-1}, y_n - r_{n,n-1}\right], \tag{1.A50}$$

[13] By definition, $\theta_N = 0$ since there are no uninformed orders after the final trading period.

Also, consider the unconditional covariance matrix $\boldsymbol{\Sigma_0}$ in (1.1), and define the corresponding conditional covariance matrix $\boldsymbol{\Sigma_n}$ for all $n \in \{1, \ldots, N-1\}$ as

$$\boldsymbol{\Sigma_n} \equiv \begin{pmatrix} \underset{(1\times 1)}{\Sigma_n^v} & \underset{(1\times M)}{\Sigma_n^{v,\boldsymbol{s}}} & \underset{(1\times N)}{\Sigma_n^{v,\boldsymbol{u}}} \\ \underset{(M\times 1)}{\Sigma_n^{\boldsymbol{s},v}} & \underset{(M\times M)}{\Sigma_n^{\boldsymbol{s}}} & \underset{(M\times N)}{\Sigma_n^{\boldsymbol{s},\boldsymbol{u}}} \\ \underset{(N\times 1)}{\Sigma_n^{\boldsymbol{u},v}} & \underset{(N\times M)}{\Sigma_n^{\boldsymbol{u},\boldsymbol{s}}} & \underset{(N\times N)}{\Sigma_n^{\boldsymbol{u}}} \end{pmatrix}, \tag{1.A51}$$

where (co)variances are conditional on the aggregate order flow history $y_{1:n}$. Then the property of multivariate Gaussian distributions implies that the conditional covariance matrix $\boldsymbol{\Sigma_n}$ can be expressed recursively as

$$\boldsymbol{\Sigma_n} = \boldsymbol{\Sigma_{n-1}} - \begin{pmatrix} \Sigma_{n-1}^{v,y_n} \\ \Sigma_{n-1}^{s_{1,0},y_n} \\ \vdots \\ \Sigma_{n-1}^{s_{M,0},y_n} \\ \Sigma_{n-1}^{u_1,y_n} \\ \vdots \\ \Sigma_{n-1}^{u_N,y_n} \end{pmatrix} \left(\Sigma_{n-1}^{y_n}\right)^{-1} \begin{pmatrix} \Sigma_{n-1}^{v,y_n} \\ \Sigma_{n-1}^{s_{1,0},y_n} \\ \vdots \\ \Sigma_{n-1}^{s_{M,0},y_n} \\ \Sigma_{n-1}^{u_1,y_n} \\ \vdots \\ \Sigma_{n-1}^{u_N,y_n} \end{pmatrix}', \tag{1.A52}$$

where the (co)variances in Proposition 1.1 are obtained after simplification. □

1.A.7 Algorithm

The equilibrium in Proposition 1.1 is solved via backward induction. Specify the number of informed traders M, the number of trading periods N, the desired period zero (co)variances Σ_0^v, $\Sigma_0^{v,s_{i,0}}$, Σ_0^{v,u_n}, $\Sigma_0^{s_{i,0},u_n}$, $\Sigma_0^{s_{i,0}}$, $\Sigma_0^{u_n}$, and the constants χ_s and χ_u for computing $\Sigma_0^{s_{i,0},s_{j,0}}$ and $\Sigma_0^{u_n,u_{n+1}}$ from (1.14) and (1.15), respectively. Note that the desired specification is only valid if the covariance matrix $\boldsymbol{\Sigma_0}$ in (1.1) is positive definite.

The algorithm starts in the last trading period N and takes as inputs guesses for the period $N-1$ conditional (co)variances $\Sigma_{N-1}^{v,s_{i,0}}$, Σ_{N-1}^{v,u_N}, $\Sigma_{N-1}^{s_{i,0},u_N}$, $\Sigma_{N-1}^{s_{i,0}}$, and

$\Sigma^{u_N}_{N-1}$.[14] Immediately, $\Sigma^{s_{i,0},s_{j,0}}_{N-1}$ follows from (1.14). By definition, the value function coefficients $a_N, \ldots, k_N$ and the projection coefficient θ_N are zero in the last trading period, simplifying the equations for β_N and λ_N to

$$\beta_N = \frac{\eta_N - \lambda_N \psi_N}{\lambda_N[2 + (M-1)\phi_N]}, \tag{1.A53}$$

$$\lambda_N = \frac{M\beta_N \Sigma^{v,s_{i,0}}_{N-1} + \Sigma^{v,u_N}_{N-1}}{M\beta_N^2 \left[\Sigma^{s_{i,0}}_{N-1} + (M-1)\Sigma^{s_{i,0},s_{j,0}}_{N-1}\right] + \Sigma^{u_N}_{N-1} + 2M\beta_N \Sigma^{s_{i,0},u_N}_{N-1}}. \tag{1.A54}$$

Rearranging yields a quadratic equation in λ_N, where only the positive root satisfies the second-order condition in period N. Given λ_N, one can compute β_N, ζ_N, γ_N, α_N, δ_N, and the value function coefficients $a_{N-1}, \ldots, k_{N-1}$.

In the induction step, the algorithm takes the following quantities as inputs in every period $n \in \{1, \ldots, N-1\}$:

(i) the projection coefficient β_{n+1},

(ii) the conditional (co)variances $\Sigma^{v,s_{i,0}}_n$, $\Sigma^{v,u_{n+1}}_n$, $\Sigma^{s_{i,0},u_{n+1}}_n$, $\Sigma^{s_{i,0}}_n$, and $\Sigma^{u_{n+1}}_n$,

(iii) and the value function coefficients $a_n, \ldots, k_n$.

With these inputs, one can simultaneously solve for β_n, $\Sigma^{v,s_{i,0}}_{n-1}$, Σ^{v,u_n}_{n-1}, $\Sigma^{s_{i,0},u_n}_{n-1}$, $\Sigma^{s_{i,0}}_{n-1}$, and $\Sigma^{u_n}_{n-1}$, resulting in a polynomial system with six equations and six unknowns. Given χ_s and χ_u, one also obtains $\Sigma^{s_{i,0},s_{j,0}}_{n-1}$ and $\Sigma^{u_n,u_{n+1}}_{n-1}$ from (1.14) and (1.15), respectively. These results allow one to compute λ_n, ζ_n, θ_n, γ_n, α_n, δ_n, and the value function coefficients $a_{n-1}, \ldots, k_{n-1}$ and proceed to the next iteration.

After the final iteration, one can compute Σ^v_n for every period $n \in \{1, \ldots, N-1\}$ since Σ^v_0 was initially specified. If

(i) the second-order condition is satisfied in every period,

(ii) conditional (co)variances are consistent in every period,

(iii) and the value function monotonically decreases over time,

[14] This paper does not compute conditional (co)variances for the last period N since these quantities are realized after the trading game.

an equilibrium was found. Additionally, if $\Sigma_0^{v,s_{i,0}}$, Σ_0^{v,u_n}, $\Sigma_0^{s_{i,0},u_n}$, $\Sigma_0^{s_{i,0}}$, and $\Sigma_0^{u_n}$ satisfy the initial specification, the algorithm terminates. Otherwise, the guesses for the period $N-1$ conditional (co)variances are adjusted, and the algorithm runs anew.

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2 Strategic Trading when the Market Maker Has a Monopoly on Short-Lived Information

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Abstract

This paper develops a strategic trading model in which the market maker has a monopoly on short-lived information. Given that modern market makers are high-frequency traders, it is assumed that market makers can process any short-lived information event faster than other traders. Since the market maker's information is short-lived, informed traders sequentially learn about it and adjust their strategies accordingly. The correlation structure of the market maker's short-lived information significantly affects the dynamic trading equilibrium. The model can generate new stylized facts like negative trading intensities and increasing price sensitivities to news.

Keywords: Kyle model, strategic trading, long-lived information, short-lived information, signal processing, price impact, market microstructure.

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2.1 Introduction

Strategic trading models can explain how information gets incorporated into asset prices. Most models assume a financial market with a single information event solely observed by informed traders. Through dynamic trading, market makers partially learn about informed traders' private information and adjust their prices accordingly.

However, empirical financial markets are characterized by the arrival of news. Any asset-specific news released during the trading day directly impacts that asset's price. Thus, the assumption of a single non-public information event may be limiting in describing trading dynamics. A natural question is what types of traders can react to news arrivals first. Since modern market makers are high-frequency traders competing for speed, a reasonable assumption is that market makers can trade news faster than other market participants.

This paper studies a strategic trading model in the spirit of Kyle (1985) and its vast extensions.[1] In the literature, the market maker can only learn about the asset value by observing aggregate order flows. Exceptions to that assumption include Jain and Mirman (1999), Daher, Mirman, and Saleeby (2014), and Foucault, Hombert, and Roşu (2016). Jain and Mirman (1999) study a one-period Kyle (1985) model in which the market maker observes the aggregate order flow and a stochastic signal about the asset value.[2] Daher, Mirman, and Saleeby (2014) extend the Jain and Mirman (1999) model to two periods. Both papers conclude that prices are more efficient if the market maker observes an additional signal next to the aggregate order flow.

[1] Notable extensions of Kyle (1985) include but are not limited to Admati and Pfleiderer (1988), Kyle (1989), Seppi (1990), Foster and Viswanathan (1990), Holden and Subrahmanyam (1992), Back (1992), Holden and Subrahmanyam (1994), Caballe and Krishnan (1994), Foster and Viswanathan (1994), Foster and Viswanathan (1996), Jain and Mirman (1999), Back, Cao, and Willard (2000), Back and Baruch (2004), Bernhardt and Miao (2004), Pasquariello (2007), Dridi and Germain (2009), Noh and S. Choi (2009), Nishide (2009), Colla and Mele (2010), Ostrovsky (2012), Rostek and Weretka (2012), Daher, Mirman, and Saleeby (2014), Collin-Dufresne and Fos (2016), Foucault, Hombert, and Roşu (2016), Lambert, Ostrovsky, and Panov (2018), J. H. Choi, Larsen, and Seppi (2019), and Sastry and Thompson (2019).

[2] Lambert, Ostrovsky, and Panov (2018) provide a most general analysis of the one-period Kyle (1985) model, where the model by Jain and Mirman (1999) is obtained as a special case.

However, empirical signals like news arrive frequently rather than only once or twice during the trading day. Since news can have substantial impacts on trading strategies and asset prices, a dynamic model is more suitable to describe a market in which the market maker can trade news faster than other traders.

Foucault, Hombert, and Roşu (2016) analyze a continuous-time version of the Kyle (1985) model where public signals arrive deterministically over time.[3] The authors distinguish between the two cases where the informed trader either "slower" or "faster" than the market maker. Clearly, the market maker has a monopoly on short-lived information if the informed trader is slower than the market maker.

However, while Foucault, Hombert, and Roşu (2016) focus on trading speed, this study focuses on the informativeness of news when the market maker is faster than informed traders. Moreover, Foucault, Hombert, and Roşu (2016) consider a single informed trader in a continuous-time setting. In contrast, this paper studies a dynamic discrete-time model with multiple informed traders who possess heterogeneous information about the asset value. Consequently, this paper accounts for game-theoretical aspects that multiple informed traders must consider when they are collectively slower than the market maker.

Building on the framework by Foster and Viswanathan (1996), the contribution to the literature lies in allowing the market maker to have a monopoly on short-lived information in a dynamic discrete-time setting. Specifically, it is assumed that the market maker sequentially receives short-lived signals that are correlated with the asset value, informed traders' long-lived private information, and future short-lived signals.

One can think of short-lived signals as news arrivals, which the market maker can process faster than other traders. Since information is short-lived, the market maker can only trade it for one period before it becomes public knowledge. Moreover, since short-lived information events are autocorrelated, informed traders and

[3] Foucault, Hombert, and Roşu (2016) also study the case where public signals arrive stochastically over time. The authors conclude that their main results remain unaffected by this change.

the market maker need to forecast future news arrivals. Overall, the model allows for two types of information events in the market: one long-lived information event solely observed by informed traders before the start of the trading game and short-lived information events that arrive sequentially over time.

Numerical results show that the correlation structure of the market maker's short-lived information significantly affects market liquidity and price efficiency. The more informative news arrivals are about the asset value, the higher the corresponding price impact. Since the market maker only prices the innovation in short-lived information, the first news arrival generally has the highest price impact. In particular, short-lived information events help the market maker extract informed traders' long-lived private information from aggregate order flows, resulting in high price discovery.

Surprisingly, depending on the correlation structure of news arrivals, price sensitivity to news innovations can also be monotonically increasing or U-shaped. In particular, prices become more sensitive to news innovations in later periods if news arrivals are (conditionally) negatively autocorrelated. That scenario is equivalent to contradicting news arriving throughout the trading day. As a result, the market maker smooths her incorporation of news into prices, resulting in the final news innovation having the highest price impact.

Finally, the model can produce new stylized facts like negative trading intensities that become positive throughout the trading day. Under specific correlation structures, equilibria are attainable in which the conditional correlation between informed traders' long-lived information and the asset value is initially negative but becomes positive over time. For example, if informed traders receive the signal that the asset value is currently undervalued, then it is optimal for them to initially sell the asset and only buy it in later trading periods. Intuitively, informed traders realize that their initial private information is biased, which is why they trade against it. However, as informed traders learn about the market maker's short-lived signals over time, they eventually switch trading directions since every subsequent news arrival reduces informed traders' initial bias.

The idea of a strategic trading model with short-lived information is not a new one. Admati and Pfleiderer (1988) consider a market in which traders can acquire short-lived information at a cost so that traders decide in every period whether to become informed or not. In contrast, Noh and S. Choi (2009) assume that informed traders have a monopoly on long-lived and short-lived information. Consequently, both papers implicitly assume that informed traders can trade short-lived information events faster than other traders in the market. That assumption may be appropriate if informed traders are high-frequency traders with faster algorithms than market makers.

Nishide (2009) studies a continuous-time Kyle (1985) model with a continuous public signal that is correlated with the asset value and uninformed orders. The underlying idea is that uninformed traders base their orders on public news. However, in contrast to this paper, the market maker in Nishide (2009) does not have a monopoly on short-lived information since news gets released continuously, eliminating the market maker's informational advantage that she enjoys in a discrete-time setting.

Finally, this paper's model derivation is closely related to Gounas (2021), who develops a strategic trading model where uninformed orders exhibit a general correlation structure. As a result, there are two types of signals in Gounas (2021): informed traders' signals that are realized before the start of the trading game and sequentially realized signals in the form of the uninformed order flow. By simply substituting the sequentially realized signals with the market maker's short-lived information, one obtains this paper's model framework.

Even though many results in Gounas (2021) apply to this paper's model, the economics differ considerably. First, in Gounas (2021), the sequentially realized signals (in the form of the uninformed order flow) remain stochastic throughout the trading day. In contrast, this paper's sequentially realized signals (in the form of the market maker's short-lived information) become public knowledge over time, allowing informed traders to incorporate those signals into their trading strategies. Second, in Gounas (2021), the market maker only observes aggregate order flows throughout the trading day. In contrast, this paper's market maker also receives short-lived private information about the asset value in every period. Consequently, this paper's market maker is also an informed trader.

In what follows, Section 2.2 sets up the model. Section 2.3 explains the equilibrium concept, and Section 2.4 derives necessary and sufficient conditions for its existence. Finally, Section 2.5 numerically evaluates the model, and Section 2.6 concludes the paper.

2.2 The Model

The model framework is based on Foster and Viswanathan (1996). Consider a market with a single risky asset. Denote the asset's stochastic liquidation value as v and its unconditional variance as $\Sigma_0^v \equiv \mathrm{Var}[v]$. Buy and sell orders for the asset can be submitted over $N \in \mathbb{N}$ equally spaced batch auctions that take place during the time interval $[0, 1]$. Strategic agents trading the asset include $I \in \mathbb{N}$ risk-neutral informed traders and a competitive market maker. Also, there are uninformed traders whose orders are assumed to arrive exogenously over time.

For all informed traders $i \in \{1, \ldots, I\}$, denote informed trader i's long-lived private signal in period zero as $s_{i,0}$. Let $\boldsymbol{s_I} \equiv [s_{1,0}, \ldots, s_{I,0}]'$ be informed traders' collective long-lived signal vector, and define the covariance matrix of $\boldsymbol{s_I}$ as $\Sigma_0^{\boldsymbol{s_I}}$, consisting of diagonals $\Sigma_0^{s_{i,0}} \equiv \mathrm{Var}[s_{i,0}]$ and off-diagonals $\Sigma_0^{s_{i,0},s_{j,0}} \equiv \mathrm{Cov}[s_{i,0}, s_{j,0}]$ where $j \in \{1, \ldots, I\}$ and $j \neq i$. Moreover, long-lived signals are correlated with the asset value, and the corresponding covariance is given by $\Sigma_0^{v,s_{i,0}} \equiv \mathrm{Cov}[v, s_{i,0}]$. Following Foster and Viswanathan (1996), assume that $\Sigma_0^{s_{i,0}}$, $\Sigma_0^{s_{i,0},s_{j,0}}$, and $\Sigma_0^{v,s_{i,0}}$ are independent of i and j for all $i, j \in \{1, \ldots, I\}$ where $i \neq j$. Thus, (co)variances are identical across all informed traders, which is a necessary assumption to resolve the underlying dimensionality problem of the dynamic trading game.

The innovation of this paper lies in allowing the market maker not only to observe aggregate order flows but also short-lived signals that are correlated with the asset value, long-lived signals, and future short-lived signals. Specifically, assume that the market maker receives a short-lived private signal $s_{M,n}$ about the asset's true value in every period $n \in \{1, \ldots, N\}$. Short-lived signals can be interpreted as news arriving throughout the trading day, where it is assumed that the market maker can trade news faster than informed traders.[4] Since the market maker's signals are short-lived, they become sequentially public information over time, that is, each signal $s_{M,n}$ gets publicly announced after trading period n.

Let $\boldsymbol{s_M} \equiv [s_{M,1}, \ldots, s_{M,n}]'$ be the market maker's collective short-lived signal vector, and define the covariance matrix of $\boldsymbol{s_M}$ as $\Sigma_0^{\boldsymbol{s_M}}$, consisting of diagonals $\Sigma_0^{s_{M,n}} \equiv \mathrm{Var}[s_{M,n}]$ and off-diagonals $\Sigma_0^{s_{M,n},s_{M,m}} \equiv \mathrm{Cov}[s_{M,n}, s_{M,m}]$ where $m \in \{1, \ldots, N\}$ and $m \neq n$. Moreover, short-lived signals are correlated with the asset value and long-lived signals. The corresponding covariances are given by $\Sigma_0^{v,s_{M,n}} \equiv \mathrm{Cov}[v, s_{M,n}]$ and $\Sigma_0^{s_{i,0},s_{M,n}} \equiv \mathrm{Cov}[s_{i,0}, s_{M,n}]$ for all $i \in \{1, \ldots, I\}$. As before, to resolve the dimensionality problem of the dynamic trading game, assume that $\Sigma_0^{s_{M,n}}$, $\Sigma_0^{s_{M,n},s_{M,m}}$, $\Sigma_0^{v,s_{M,n}}$, and $\Sigma_0^{s_{i,0},s_{M,n}}$ are independent of i, n, and m for all $i \in \{1, \ldots, I\}$ and $n, m \in \{1, \ldots, N\}$ where $n \neq m$.

Summing up, informed traders have a monopoly on long-lived information, while the market maker has a monopoly on short-lived information. Since the market maker possesses short-lived private information about the asset value in every period, the market maker is also an informed trader. However, note that the market maker never fully learns about informed traders' signals, whereas informed traders perfectly learn about the market maker's signals over time.

The central assumption of this paper is that the asset value, long-lived signals,

[4]This assumption is justified by the fact that modern market makers are high-frequency traders who can react to news arrivals faster than other traders.

and short-lived signals follow a multivariate normal distribution:[5]

$$\begin{pmatrix} v \\ \boldsymbol{s_I} \\ \boldsymbol{s_M} \end{pmatrix} \sim \mathcal{N}\left[\begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} \underset{(1\times 1)}{\Sigma_0^{v}} & \underset{(1\times I)}{\Sigma_0^{v,\boldsymbol{s_I}}} & \underset{(1\times N)}{\Sigma_0^{v,\boldsymbol{s_M}}} \\ \underset{(I\times 1)}{\Sigma_0^{\boldsymbol{s_I},v}} & \underset{(I\times I)}{\Sigma_0^{\boldsymbol{s_0}}} & \underset{(I\times N)}{\Sigma_0^{\boldsymbol{s_I},\boldsymbol{s_M}}} \\ \underset{(N\times 1)}{\Sigma_0^{\boldsymbol{s_M},v}} & \underset{(N\times I)}{\Sigma_0^{\boldsymbol{s_M},\boldsymbol{s_I}}} & \underset{(N\times N)}{\Sigma_0^{\boldsymbol{s_M}}} \end{pmatrix}\right], \tag{2.1}$$

where $\Sigma_0^{v,\boldsymbol{s_I}}$ has entries $\Sigma_0^{v,s_{i,0}}$, $\Sigma_0^{v,\boldsymbol{s_M}}$ has entries $\Sigma_0^{v,s_{M,n}}$, and $\Sigma_0^{\boldsymbol{s_I},\boldsymbol{s_M}}$ has entries $\Sigma_0^{s_{i,0},s_{M,n}}$. Informed traders and the market maker know the distribution (2.1) and form beliefs about v, $\boldsymbol{s_I}$, and $\boldsymbol{s_M}$ during the trading day.

Similar to Foster and Viswanathan (1996), the model provides a sufficient statistic that will facilitate the forecasting problem of the dynamic trading game.

Lemma 2.1. *Assume the distribution assumption (2.1) holds, then*

$$\mathbb{E}\left[v \middle| \boldsymbol{s_I}, \boldsymbol{s_M}\right] = \theta_I \sum_{i=1}^{I} s_{i,0} + \theta_M \sum_{t=1}^{N} s_{M,t} \equiv \hat{v}, \tag{2.2}$$

where θ_I and θ_M are constants.

Proof. See Appendix 2.A.1.

Lemma 2.1 implies that instead of forecasting the asset value v, it is sufficient to forecast the weighted sum of both informed traders' and the market maker's signals $\hat{v}$. Intuitively, $\hat{v}$ captures all available information about the asset value in the market. Consequently, the best forecast of the asset value is the combined forecast of all signals correlated with the asset value, where the corresponding correlations are captured by the constants θ_I and θ_M.

[5] The distribution assumption (2.1) is similar to the one in Gounas (2021). In Gounas (2021), it is assumed that uninformed orders (instead of the market maker's signals) are correlated with the asset value, informed traders' signals, and each other over time. While the distributions are statistically equivalent, the economics differ considerably. Informed traders and the market maker never fully learn about uninformed orders since they are hidden in aggregate order flows. In contrast, there is no uncertainty about the market maker's signals at the end of the trading day since they become public information over time. Moreover, this paper's market maker is also an informed trader since she possesses short-lived private information about the asset value in every period. As a result, this paper's strategies and beliefs will differ from those in Gounas (2021).

Finally, uninformed traders submit the aggregate quantity u_n in every period $n \in \{1, \ldots, N\}$. It is assumed that uninformed orders arrive identically and independently distributed over time with zero mean and homogeneous variance $\Sigma_0^u \equiv \mathrm{Var}[u]$. As discussed in Gounas (2021), this is an unreasonable assumption since it implies aggregate order flows to be unpredictable, standing in contrast to the significant autocorrelation found in empirical order flows.[6] However, this assumption is kept for simplicity and may be addressed in future research.

Define the aggregate order flow y_n in every period $n \in \{1, \ldots, N\}$ as

$$y_n \equiv \sum_{i=1}^{I} x_{i,n} + u_n, \tag{2.3}$$

where $x_{i,n}$ is informed trader i's order in period n. The market maker observes y_n and needs to set the clearing price p_n in every period $n \in \{1, \ldots, N\}$. Due to perfect competition, she makes zero expected profits, implying

$$p_n \equiv \mathbb{E}\left[v \middle| s_{M,1:n}, y_{1:n}\right], \tag{2.4}$$

where the market maker's information set in trading round n consists of the histories $s_{M,1:n} \equiv (s_{M,1}, \ldots, s_{M,n})$ and $y_{1:n} \equiv (y_1, \ldots, y_n)$. Interpreting short-lived signals as news arrivals, the perfect competition condition (2.4) creates a direct link between news and asset prices.

The Bellman principle of optimality implies that every informed trader $i \in \{1, \ldots, I\}$ maximizes the following objective function in each period $n \in \{1, \ldots, N\}$:

$$\max_{x_{i,n}} \mathbb{E}\left[(v - p_n)x_{i,n} + V_{i,n} \middle| s_{i,0}, s_{M,1:n-1}, y_{1:n-1}, \tilde{x}_{i,1:n-1}\right], \tag{2.5}$$

where

$$V_{i,n} \equiv \max_{x_{i,n+1},\ldots,x_{i,N}} \mathbb{E}\left[\sum_{m=n+1}^{N} (v - p_m)x_{i,m} \middle| s_{i,0}, s_{M,1:n}, y_{1:n}, \tilde{x}_{i,1:n}\right] \tag{2.6}$$

[6] See Hasbrouck (1991a,b) and Brogaard, Hendershott, and Riordan (2019).

is informed trader i's value function after n trading rounds. Thus, in each period, every informed trader maximizes his terminal expected profit conditional on his available information. Note that every informed trader sequentially learns about the market maker's signals and incorporates them into his profit maximization problem. Moreover, by definition of the Bellman equation (2.5), informed trader i's strategy $x_{i,n}$ in period n must be optimal for any arbitrary trading history $\tilde{x}_{i,1:n-1} \equiv (\tilde{x}_{i,1}, \ldots, \tilde{x}_{i,n-1})$. Therefore, optimal trading strategies must also account for suboptimal play in the past.

Given informed traders' and the market maker's objective functions, a dynamic trading equilibrium is defined as in Foster and Viswanathan (1996).

Definition 2.1. *A Bayesian Nash equilibrium is defined by the set of strategies* $\{x_{1,n}, \ldots, x_{I,n}, p_n\}_{n=1}^{N}$ *if the following two conditions hold:*

(i) The Bellman equation (2.5) is maximized for all informed traders $i \in \{1, \ldots, I\}$ *in every period* $n \in \{1, \ldots, N\}$.

(ii) The clearing price satisfies (2.4) in every period $n \in \{1, \ldots, N\}$.

Note that Definition 2.1 implies that there cannot exist a Bayesian Nash equilibrium in which a competitive market maker possesses long-lived private information, and the proof is shown by contradiction. Suppose the market maker receives a long-lived private signal $s_{M,1}$ in the first period. In that case, the pricing rule (2.4) implies that the price p_1 in the first period is a function of the signal $s_{M,1}$ and the aggregate order flow y_1. However, both p_1 and y_1 become public information after the first round of trading. Since Definition 2.1 requires the functional form of the market maker's pricing rule to be common knowledge, informed traders can derive the market maker's signal $s_{M,1}$ from observing p_1 and y_1. Consequently, $s_{M,1}$ cannot be long-lived private information. By induction, it follows that a competitive market maker cannot possess long-lived private information in a Bayesian Nash equilibrium.

2.3 The Conjectured Equilibrium

This paper focuses on a linear and symmetric trading equilibrium. Let $s_{i,n-1} \equiv s_{i,0} - \mathbb{E}\left[s_{i,0} \middle| s_{M,1:n-1}, y_{1:n-1}\right]$ be informed trader i's residual long-lived signal after $n-1$ trading rounds, and conjecture that informed trader i's equilibrium strategy $x_{i,n}$ in trading period n is linear in $s_{i,n-1}$:

$$x_{i,n} = \beta_n s_{i,n-1}, \tag{2.7}$$

for all informed traders $i \in \{1, \ldots, I\}$ in each period $n \in \{1, \ldots, N\}$. As in Foster and Viswanathan (1996), the trading strategy (2.7) implies that the market maker cannot predict whether informed traders will submit buy or sell orders in the next trading period, yielding the following lemma.

Lemma 2.2. *Given (2.7), the market maker's prices and beliefs satisfy the following recursions:*

$$p_n \equiv \mathbb{E}\left[v \middle| s_{M,1:n}, y_{1:n}\right] = p_{n-1} + \mu_n(s_{M,n} - r_{n,n-1}) + \lambda_n y_n, \tag{2.8}$$

$$t_{i,n} \equiv \mathbb{E}\left[s_{i,0} \middle| s_{M,1:n}, y_{1:n}\right] = t_{i,n-1} + \alpha_n(s_{M,n} - r_{n,n-1}) + \zeta_n y_n, \tag{2.9}$$

$$r_{n+t,n} \equiv \mathbb{E}\left[s_{M,n+t} \middle| s_{M,1:n}, y_{1:n}\right] = r_{n,n-1} + \omega_n(s_{M,n} - r_{n,n-1}) + \delta_n y_n, \tag{2.10}$$

for all periods $n \in \{1, \ldots, N\}$, informed traders $i \in \{1, \ldots, I\}$, and forecasting horizons $t \in \{1, \ldots, N-n\}$. Moreover, the multivariate projection coefficients μ_n, λ_n, α_n, ζ_n, ω_n, and δ_n satisfy the relations

$$\mu_n = I\theta_I \alpha_n + \theta_M \Big[1 + (N-n)\omega_n\Big], \tag{2.11}$$

$$\lambda_n = I\theta_I \zeta_n + (N-n)\theta_M \delta_n. \tag{2.12}$$

Proof. See Appendix 2.A.2.

According to Lemma 2.2, the market maker's prices and beliefs are not only linear functions of the aggregate order flow y_n but also of the innovation in short-lived information $s_{M,n} - r_{n,n-1}$. Note that the projection coefficient λ_n measures the price sensitivity to aggregate order flows and is equivalent to an illiquidity parameter as in Kyle (1985). However, since projection coefficients are multivariate now, the market maker also considers interaction effects that aggregate order flows have with short-lived signal innovations.

Of particular interest is the projection coefficient μ_n, which measures the impact of short-lived signal innovations on prices and can be thought of as a news parameter. The larger μ_n, the more sensitive prices are to news innovations. Moreover, instead of forecasting all future short-lived signals, (2.10) implies that the market maker only needs to forecast the next period's short-lived signal $s_{M,n+1}$ after n rounds of trading. This result follows from the symmetry conditions imposed for (co)variances, which resolve the dimensionality issue of the dynamic trading game.

Finally, Lemma 2.2 and the law of iterated expectations imply that the price p_n in trading period $n \in \{1, \dots, N\}$ can be expressed as[7]

$$p_n = I\theta_I t_{i,n} + \theta_M \left[\sum_{t=1}^{n} s_{M,t} + (N-n) r_{n+1,n} \right]. \tag{2.13}$$

Consequently, the market maker's price in period n is a weighted sum of her belief about informed traders' long-lived signal $t_{i,n}$, her realized short-lived signals $s_{M,1:n}$, and her belief about future short-lived signals $r_{n+1,n}$. This result will facilitate informed traders' dynamic programming problem since prices will become a redundant state variable.

To measure information efficiency during the trading day, define for all periods $n \in \{1, \dots, N\}$ and all informed traders $i, j \in \{1, \dots, I\}$:

$$\begin{aligned}
\Sigma_n^{\hat{v}} &\equiv \mathrm{Var}\left[\hat{v} \middle| s_{M,1:n}, y_{1:n}\right], & \Sigma_n^{s_{i,0}} &\equiv \mathrm{Var}\left[s_{i,0} \middle| s_{M,1:n}, y_{1:n}\right], \\
\Sigma_n^{s_{M,n+1}} &\equiv \mathrm{Var}\left[s_{M,n+1} \middle| s_{M,1:n}, y_{1:n}\right], & \Sigma_n^{s_{i,0}, s_{j,0}} &\equiv \mathrm{Cov}\left[s_{i,0}, s_{j,0} \middle| s_{M,1:n}, y_{1:n}\right], \\
\Sigma_n^{s_{i,0}, s_{M,n+1}} &\equiv \mathrm{Cov}\left[s_{i,0}, s_{M,n+1} \middle| s_{M,1:n}, y_{1:n}\right], & \Sigma_n^{s_{M,n+1}, s_{M,n+2}} &\equiv \mathrm{Cov}\left[s_{M,n+1}, s_{M,n+2} \middle| s_{M,1:n}, y_{1:n}\right].
\end{aligned}$$

[7] This result is shown in Appendix 2.A.2.

These conditional (co)variances capture the market maker's remaining uncertainty about long-lived and future short-lived signals in every trading period. In particular, they satisfy the following relations.

Lemma 2.3. *Given (2.7), the market maker's conditional (co)variance updates of long-lived and future short-lived signals satisfy*

$$\Sigma_{n-1}^{s_{i,0}} - \Sigma_{n}^{s_{i,0}} = \Sigma_{n-1}^{s_{i,0},s_{j,0}} - \Sigma_{n}^{s_{i,0},s_{j,0}}, \tag{2.14}$$

$$\Sigma_{n-1}^{s_{M,n}} - \Sigma_{n}^{s_{M,n+1}} = \Sigma_{n-1}^{s_{M,n},s_{M,n+1}} - \Sigma_{n}^{s_{M,n+1},s_{M,n+2}}, \tag{2.15}$$

for all periods $n \in \{1, \dots, N-1\}$ and informed traders $i, j \in \{1, \dots, I\}$. Moreover, it holds that

$$\begin{aligned}\Sigma_n^{\hat{v}} = I\theta_I^2\Big[\Sigma_n^{s_{i,0}} + (I-1)\Sigma_n^{s_{i,0},s_{j,0}}\Big] + 2I(N-n)\theta_I\theta_M\Sigma_n^{s_{i,0},s_{M,n+1}} \\ + (N-n)\theta_M^2\Big[\Sigma_n^{s_{M,n+1}} + (N-n-1)\Sigma_n^{s_{M,n+1},s_{M,n+2}}\Big].\end{aligned} \tag{2.16}$$

Proof. See Appendix 2.A.3.

From (2.14) and (2.15), it can be concluded that

$$\Sigma_{n-1}^{s_{i,0}} - \Sigma_{n-1}^{s_{i,0},s_{j,0}} \equiv \chi_I, \tag{2.17}$$

$$\Sigma_{n-1}^{s_{M,n}} - \Sigma_{n-1}^{s_{M,n},s_{M,n+1}} \equiv \chi_M, \tag{2.18}$$

for all periods $n \in \{1, \dots, N-1\}$, where χ_I and χ_M are constants. Therefore, similar to Gounas (2021), the difference between variances and covariances for long-lived and future short-lived signals remains constant over time.

Finally, recall from Lemma 2.1 that $\hat{v}$ is defined as the conditional expectation of the asset value given all available information in the market. Thus, $\Sigma_n^{\hat{v}}$ in (2.16) is the market maker's conditional variance of the best forecast of the asset value. Since the market maker cannot perfectly learn the asset value in discrete time,[8] it must hold that $\Sigma_N^{\hat{v}} > 0$, yielding the same lower bounds for $\Sigma_N^{s_{i,0}}$ and $\Sigma_N^{s_{i,0},s_{j,0}}$ as

[8] See Foster and Viswanathan (1996) for proof of this statement.

in Foster and Viswanathan (1996):

$$\Sigma_N^{s_{i,0}} > \frac{I-1}{I}\chi_I, \tag{2.19}$$

$$\Sigma_N^{s_{i,0},s_{j,0}} > -\frac{1}{I}\chi_I, \tag{2.20}$$

for all informed traders $i, j \in \{1, \ldots, I\}$ where $i \neq j$. From (2.14) and (2.20), it follows that the conditional correlation between informed traders' long-lived signals monotonically decreases over time and must eventually be negative for a sufficiently large number of trading periods.

2.4 Informed Traders' Updating Processes

The Bellman equation (2.5) states that informed traders' trading strategies must be optimal for any arbitrary trading history. For this reason, there is a need to distinguish between equilibrium and off-equilibrium play. First, it is shown how informed traders update their beliefs along the equilibrium path. Then the updating processes are extended to account for off-equilibrium play. Finally, this section concludes with the necessary and sufficient conditions for equilibrium.

2.4.1 Updating along the Equilibrium Path

Lemma 2.2 implies that the trading strategy (2.7), which will be played in equilibrium, can be written as

$$\begin{aligned} x_{i,n} = \beta_n s_{i,n-1} = \beta_n(s_{i,0} - t_{i,n-1}) &= \beta_n \left(s_{i,0} - \sum_{t=1}^{n-1} \Big[\alpha_t(s_{M,t} - r_{t,t-1}) + \zeta_t y_t \Big] \right) \\ &= \beta_n \left(s_{i,0} - \sum_{t=1}^{n-1} \left[\alpha_t \left(s_{M,t} - \sum_{r=1}^{t-1} \left(\prod_{s=r+1}^{t-1} \Big[1 - \omega_s\Big] \right) \Big[\omega_r s_{M,r} + \delta_r y_r\Big] \right) + \zeta_t y_t \right] \right), \end{aligned} \tag{2.21}$$

for all informed traders $i \in \{1, \ldots, I\}$ in each period $n \in \{1, \ldots, N\}$. Therefore, informed trader i's equilibrium strategy in period n only depends on his individual long-lived signal $s_{i,0}$, the public history of short-lived signals $s_{M,1:n-1}$, and the public history of aggregate order flows $y_{1:n-1}$. Most importantly, the equilibrium strategy is independent of the individual trading history $x_{i,1:n-1}$, yielding the following lemma.

Lemma 2.4. *Given (2.7), (2.8), (2.9), and (2.10), informed traders' updating processes satisfy*

$$\mathbb{E}\left[v - p_{n-1}\middle|s_{i,0}, s_{M,1:n-1}, y_{1:n-1}, x_{i,1:n-1}\right] = \eta_n s_{i,n-1}, \tag{2.22}$$

$$\mathbb{E}\left[s_{j,n-1}\middle|s_{i,0}, s_{M,1:n-1}, y_{1:n-1}, x_{i,1:n-1}\right] = \phi_n s_{i,n-1}, \tag{2.23}$$

$$\mathbb{E}\left[s_{M,n} - r_{n,n-1}\middle|s_{i,0}, s_{M,1:n-1}, y_{1:n-1}, x_{i,1:n-1}\right] = \psi_n s_{i,n-1}, \tag{2.24}$$

for all periods $n \in \{1, \ldots, N\}$ and informed traders $i, j \in \{1, \ldots, I\}$ where $i \neq j$.

Proof. See Appendix 2.A.4.

To conclude, along equilibrium play, the only state variable for every informed trader i after $n - 1$ periods is his residual long-lived signal $s_{i,n-1}$.[9]

2.4.2 Updating along Off-Equilibrium Paths

To ensure that informed traders do not have an incentive to deviate from the equilibrium strategy (2.7), one needs to account for off-equilibrium play. By deviating, any informed trader $i \in \{1, \ldots, I\}$ can distort equilibrium beliefs since deviations from (2.7) are invisible to the market maker and other informed traders.

Suppose informed trader i is the only informed trader to have deviated from the equilibrium strategy (2.7). To account for off-equilibrium play, closely follow Foster and Viswanathan (1996) and define the following equilibrium quantities for all trading periods $n \in \{1, \ldots, N\}$ and informed traders $j \in \{1, \ldots, I\}$:

$$\hat{y}_n^i \equiv \sum_{j=1}^{I} \beta_n \hat{s}_{j,n-1}^i + u_n,$$

[9] This result also applies to Foster and Viswanathan (1996) and Gounas (2021). However, since beliefs differ in each paper, so does the functional form of $s_{i,n-1}$.

$$\hat{s}^i_{j,n} \equiv s_{j,0} - \hat{t}^i_{j,n} \text{ and } \hat{s}^i_{j,0} \equiv s_{j,0},$$
$$\hat{p}^i_n \equiv \hat{p}^i_{n-1} + \mu_n \left(s_{M,n} - \hat{r}^i_{n,n-1}\right) + \lambda_n \hat{y}^i_n \text{ and } \hat{p}^i_0 \equiv 0,$$
$$\hat{t}^i_{j,n} \equiv \hat{t}^i_{j,n-1} + \alpha_n \left(s_{M,n} - \hat{r}^i_{n,n-1}\right) + \zeta_n \hat{y}^i_n \text{ and } \hat{t}^i_{j,0} \equiv 0,$$
$$\hat{r}^i_{n+1,n} \equiv \hat{r}^i_{n,n-1} + \omega_n \left(s_{M,n} - \hat{r}^i_{n,n-1}\right) + \theta_n \hat{y}^i_n \text{ and } \hat{r}^i_{1,0} \equiv 0.$$

These quantities would have been realized in equilibrium if all informed traders (including i) had played the equilibrium strategy (2.7). A fundamental result from Foster and Viswanathan (1996), which also applies to this paper, is that the above equilibrium quantities are in informed trader i's information set even along off-equilibrium paths.

Lemma 2.5. *Given (2.7), (2.8), (2.9), and (2.10), assume any informed trader $i \in \{1, \ldots, I\}$ has deviated from (2.7). Then for all periods $n \in \{1, \ldots, N\}$:*

$$\left(s_{i,0}, s_{M,1:n}, y_{1:n}, \tilde{x}_{i,1:n}\right) \equiv \left(s_{i,0}, s_{M,1:n}, \hat{y}^i_{1:n}, \tilde{x}_{i,1:n}\right), \tag{2.25}$$

where $\hat{y}^i_{1:n} \equiv (\hat{y}^i_1, \ldots, \hat{y}^i_n)$.

Proof. See Appendix 2.A.5.

According to Lemma 2.5, any informed trader i who has deviated from the equilibrium strategy (2.7) can recursively reconstruct the equilibrium order flow history and, by extension, the history of equilibrium prices and beliefs. It is this result that helps characterize off-equilibrium paths.

Lemma 2.6. *Given (2.7), (2.8), (2.9), and (2.10), assume any informed trader $i \in \{1, \ldots, I\}$ has deviated from the equilibrium strategy (2.7). Then informed trader i's updating processes satisfy*

$$\mathbb{E}\left[v - p_{n-1} \middle| s_{i,0}, s_{M,1:n-1}, y_{1:n-1}, \tilde{x}_{i,1:n-1}\right] = \eta_n \hat{s}^i_{i,n-1} + \left(\hat{p}^i_{n-1} - p_{n-1}\right), \tag{2.26}$$
$$\mathbb{E}\left[s_{j,n-1} \middle| s_{i,0}, s_{M,1:n-1}, y_{1:n-1}, \tilde{x}_{i,1:n-1}\right] = \phi_n \hat{s}^i_{i,n-1} + \left(\hat{t}^i_{j,n-1} - t_{j,n-1}\right), \tag{2.27}$$

$$\mathbb{E}\left[s_{M,n} - r_{n,n-1}\middle| s_{i,0}, s_{M,1:n-1}, y_{1:n-1}, \tilde{x}_{i,1:n-1}\right] = \psi_n \hat{s}^i_{i,n-1} + \left(\hat{r}^i_{n,n-1} - r_{n,n-1}\right), \tag{2.28}$$

for all periods $n \in \{2, \ldots, N\}$ and informed traders $j \in \{1, \ldots, I\}$ where $j \neq i$.

Proof. See Appendix 2.A.6.

Along off-equilibrium play, the state variables for every informed trader i after $n-1$ periods are his residual long-lived signal $\hat{s}^i_{i,n-1}$ and the manipulations of conditional beliefs about the asset value $\hat{p}^i_{n-1} - p_{n-1}$, long-lived signals $\hat{t}^i_{j,n-1} - t_{j,n-1}$, and the next period's short-lived signal $\hat{r}^i_{n,n-1} - r_{n,n-1}$.

However, $\hat{p}^i_{n-1} - p_{n-1}$ is a redundant state variable since (2.13) implies

$$\hat{p}^i_{n-1} - p_{n-1} = I\theta_I \left(\hat{t}^i_{j,n-1} - t_{j,n-1}\right) + (N-n+1)\theta_M \left(\hat{r}^i_{n,n-1} - r_{n,n-1}\right). \tag{2.29}$$

Therefore, informed trader i's manipulation of the market maker's conditional belief about the asset value can be reconstructed from his manipulations of the market maker's conditional beliefs about long-lived and future short-lived signals. To conclude, along off-equilibrium play, the non-redundant state variables for every informed trader i after $n-1$ periods are $\hat{s}^i_{i,n-1}$, $\hat{t}^i_{j,n-1} - t_{j,n-1}$, and $\hat{r}^i_{n,n-1} - r_{n,n-1}$.

2.4.3 Necessary and Sufficient Conditions for Equilibrium

A linear dynamic trading equilibrium implies that informed trader i's optimal trading strategy in period n must be linear in the non-redundant state variables:

$$x_{i,n} = \beta_n \hat{s}^i_{i,n-1} + \gamma_n \left(\hat{t}^i_{j,n-1} - t_{j,n-1}\right) + \rho_n \left(\hat{r}^i_{n,n-1} - r_{n,n-1}\right), \tag{2.30}$$

for all informed traders $i \in \{1, \ldots, I\}$ in each period $n \in \{1, \ldots, N\}$. Moreover, informed trader i's value function in the Bellman equation (2.5) must be quadratic in the non-redundant state variables:

$$
\begin{aligned}
V_{i,n-1} = a_{n-1}\left(\hat{s}^i_{i,n-1}\right)^2 &+ b_{n-1}\left(\hat{t}^i_{j,n-1} - t_{j,n-1}\right)^2 + c_{n-1}\left(\hat{r}^i_{n,n-1} - r_{n,n-1}\right)^2 \\
&+ d_{n-1}\hat{s}^i_{i,n-1}\left(\hat{t}^i_{j,n-1} - t_{j,n-1}\right) + e_{n-1}\hat{s}^i_{i,n-1}\left(\hat{r}^i_{n,n-1} - r_{n,n-1}\right) \\
&+ f_{n-1}\left(\hat{t}^i_{j,n-1} - t_{j,n-1}\right)\left(\hat{r}^i_{n,n-1} - r_{n,n-1}\right) + g_{n-1},
\end{aligned}
\tag{2.31}
$$

for all periods $n \in \{1, \dots, N\}$ and all informed traders $i, j \in \{1, \dots, I\}$ where $i \neq j$. Note that the equilibrium strategy (2.7) is consistent with (2.30) since $t_{j,n-1} = \hat{t}^i_{j,n-1}$ and $r_{n,n-1} = \hat{r}^i_{n,n-1}$ along equilibrium play. However, as the following proposition shows, one has to account for off-equilibrium paths to ensure that informed traders do not have an incentive to deviate from equilibrium play.

Proposition 2.1. *A Markov perfect equilibrium is defined by the strategies and beliefs (2.8), (2.9), (2.10), (2.26), (2.27), (2.28), and (2.30) if they satisfy the following system of equations for all periods $n \in \{1, \dots, N\}$:*

$$
\beta_n = \frac{\eta_n - \mu_n\psi_n - (1 - \alpha_n\psi_n)(d_n\zeta_n + e_n\delta_n)}{\lambda_n[2 + (I-1)\phi_n] - \zeta_n[1 + (I-1)\phi_n](d_n\zeta_n + e_n\delta_n)},
$$

$$
\gamma_n = \frac{I\theta_I - (I-1)\beta_n[\lambda_n - \delta_n(2c_n\delta_n + f_n\zeta_n)] - [1 - (I-1)\zeta_n\beta_n](2b_n\zeta_n + f_n\delta_n)}{2\lambda_n - \zeta_n(2b_n\zeta_n + f_n\delta_n) - \delta_n(2c_n\delta_n + f_n\zeta_n)},
$$

$$
\rho_n = \frac{(N-n+1)\theta_M - \mu_n + \alpha_n(2b_n\zeta_n + f_n\delta_n) - (1-\omega_n)(2c_n\delta_n + f_n\zeta_n)}{2\lambda_n - \zeta_n(2b_n\zeta_n + f_n\delta_n) - \delta_n(2c_n\delta_n + f_n\zeta_n)},
$$

$$
\mu_n = I\theta_I\alpha_n + \theta_M\Big[1 + (N-n)\omega_n\Big],
$$

$$
\lambda_n = I\theta_I\zeta_n + (N-n)\theta_M\delta_n,
$$

$$
\alpha_n = \frac{\Sigma^{s_{i,0},s_{M,n}}_{n-1}\Sigma^u_0}{\Sigma^{s_{M,n}}_{n-1}\left(I\beta_n^2\left[\Sigma^{s_{i,0}}_{n-1} + (I-1)\Sigma^{s_{i,0},s_{j,0}}_{n-1}\right] + \Sigma^u_0\right) - \left[I\beta_n\Sigma^{s_{i,0},s_{M,n}}_{n-1}\right]^2},
$$

$$
\zeta_n = \frac{\beta_n\left(\Sigma^{s_{M,n}}_{n-1}\left[\Sigma^{s_{i,0}}_{n-1} + (I-1)\Sigma^{s_{i,0},s_{j,0}}_{n-1}\right] - I\left[\Sigma^{s_{i,0},s_{M,n}}_{n-1}\right]^2\right)}{\Sigma^{s_{M,n}}_{n-1}\left(I\beta_n^2\left[\Sigma^{s_{i,0}}_{n-1} + (I-1)\Sigma^{s_{i,0},s_{j,0}}_{n-1}\right] + \Sigma^u_0\right) - \left[I\beta_n\Sigma^{s_{i,0},s_{M,n}}_{n-1}\right]^2},
$$

$$
\omega_n = \frac{\Sigma^{s_{M,n},s_{M,n+1}}_{n-1}\left(I\beta_n^2\left[\Sigma^{s_{i,0}}_{n-1} + (I-1)\Sigma^{s_{i,0},s_{j,0}}_{n-1}\right] + \Sigma^u_0\right) - \left[I\beta_n\Sigma^{s_{i,0},s_{M,n}}_{n-1}\right]^2}{\Sigma^{s_{M,n}}_{n-1}\left(I\beta_n^2\left[\Sigma^{s_{i,0}}_{n-1} + (I-1)\Sigma^{s_{i,0},s_{j,0}}_{n-1}\right] + \Sigma^u_0\right) - \left[I\beta_n\Sigma^{s_{i,0},s_{M,n}}_{n-1}\right]^2},
$$

$$
\delta_n = \frac{I\beta_n\Sigma^{s_{i,0},s_{M,n}}_{n-1}\left[\Sigma^{s_{M,n}}_{n-1} - \Sigma^{s_{M,n},s_{M,n+1}}_{n-1}\right]}{\Sigma^{s_{M,n}}_{n-1}\left(I\beta_n^2\left[\Sigma^{s_{i,0}}_{n-1} + (I-1)\Sigma^{s_{i,0},s_{j,0}}_{n-1}\right] + \Sigma^u_0\right) - \left[I\beta_n\Sigma^{s_{i,0},s_{M,n}}_{n-1}\right]^2},
$$

where the value function coefficients are given by the recursions

$$
\begin{aligned}
a_{n-1} &= a_n\Big(1-\alpha_n\psi_n-\zeta_n\beta_n\Big[1+(I-1)\phi_n\Big]\Big)^2+\beta_n\Big(\eta_n-\mu_n\psi_n-\lambda_n\beta_n\Big[1+(I-1)\phi_n\Big]\Big),\\
b_{n-1} &= b_n\Big(1-\zeta_n\Big[\gamma_n+(I-1)\beta_n\Big]\Big)^2+\gamma_n\Big(I\theta_I-\lambda_n\Big[\gamma_n+(I-1)\beta_n\Big]\Big)\\
&\quad+c_n\delta_n^2\Big[\gamma_n+(I-1)\beta_n\Big]^2-f_n\delta_n\Big[\gamma_n+(I-1)\beta_n\Big]\Big(1-\zeta_n\Big[\gamma_n+(I-1)\beta_n\Big]\Big),\\
c_{n-1} &= c_n(1-\omega_n-\delta_n\rho_n)^2+\rho_n\Big[(N-n+1)\theta_M-\mu_n-\lambda_n\rho_n\Big]+b_n(\alpha_n+\zeta_n\rho_n)^2\\
&\quad-f_n(\alpha_n+\zeta_n\rho_n)(1-\omega_n-\delta_n\rho_n),\\
d_{n-1} &= \gamma_n\Big(\eta_n-\mu_n\psi_n-\lambda_n\beta_n\Big[1+(I-1)\phi_n\Big]\Big)+\beta_n\Big(I\theta_I-\lambda_n\Big[\gamma_n+(I-1)\beta_n\Big]\Big)\\
&\quad+\Big(1-\alpha_n\psi_n-\zeta_n\beta_n\Big[1+(I-1)\phi_n\Big]\Big)\Big(d_n\Big(1-\zeta_n\Big[\gamma_n+(I-1)\beta_n\Big]\Big)\\
&\quad-e_n\delta_n\Big[\gamma_n+(I-1)\beta_n\Big]\Big),\\
e_{n-1} &= \rho_n\Big(\eta_n-\mu_n\psi_n-\lambda_n\beta_n\Big[1+(I-1)\phi_n\Big]\Big)+\beta_n\Big[(N-n+1)\theta_M-\mu_n-\lambda_n\rho_n\Big]\\
&\quad+\Big(1-\alpha_n\psi_n-\zeta_n\beta_n\Big[1+(I-1)\phi_n\Big]\Big)\Big[e_n(1-\omega_n-\delta_n\rho_n)-d_n(\alpha_n+\zeta_n\rho_n)\Big],\\
f_{n-1} &= \rho_n\Big(I\theta_I-\lambda_n\Big[\gamma_n+(I-1)\beta_n\Big]\Big)+\gamma_n\Big[(N-n+1)\theta_M-\mu_n-\lambda_n\rho_n\Big]\\
&\quad+\Big(1-\zeta_n\Big[\gamma_n+(I-1)\beta_n\Big]\Big)\Big[f_n(1-\omega_n-\delta_n\rho_n)-2b_n(\alpha_n+\zeta_n\rho_n)\Big]\\
&\quad+\delta_n\Big[\gamma_n+(I-1)\beta_n\Big]\Big[f_n(\alpha_n+\zeta_n\rho_n)-2c_n(1-\omega_n-\delta_n\rho_n)\Big],\\
g_{n-1} &= g_n+a_n\Bigg[\zeta_n^2\Sigma_0^u+\alpha_n^2\operatorname{Var}\Big[s_{M,n}-r_{n,n-1}\Big|s_{i,0},s_{M,1:n-1},y_{1:n-1}\Big]\\
&\quad+2(I-1)\zeta_n\beta_n\alpha_n\operatorname{Cov}\Big[s_{j,n-1},s_{M,n}-r_{n,n-1}\Big|s_{i,0},s_{M,1:n-1},y_{1:n-1}\Big]\\
&\quad+(I-1)\zeta_n^2\beta_n^2\Big(\operatorname{Var}\Big[s_{j,n-1}\Big|s_{i,0},s_{M,1:n-1},y_{1:n-1}\Big]\\
&\quad+(I-2)\operatorname{Cov}\Big[s_{j,n-1},s_{k,n-1}\Big|s_{i,0},s_{M,1:n-1},y_{1:n-1}\Big]\Big)\Bigg],
\end{aligned}
$$

with terminal condition $a_N=b_N=c_N=d_N=e_N=f_N=g_N=0$. *Also,*

$$
\begin{aligned}
\eta_n &= \theta_I\Big[1+(I-1)\phi_n\Big]+(N-n+1)\theta_M\psi_n,\\
\phi_n &= \frac{\Sigma_{n-1}^{s_{i,0},s_{j,0}}}{\Sigma_{n-1}^{s_{i,0}}},
\end{aligned}
$$

$$\psi_n = \frac{\Sigma_{n-1}^{s_{i,0},s_{M,n}}}{\Sigma_{n-1}^{s_{i,0}}},$$

and for all $i, j, k \in \{1, \ldots, I\}$ *where* $i \neq j \neq k$*, it holds that*

$$\begin{aligned}
\mathrm{Var}\left[s_{M,n} - r_{n,n-1} \middle| s_{i,0}, s_{M,1:n-1}, y_{1:n-1}\right] &= \Sigma_{n-1}^{s_{M,n}} - \psi_n^2 \Sigma_{n-1}^{s_{i,0}}, \\
\mathrm{Var}\left[s_{j,n-1} \middle| s_{i,0}, s_{M,1:n-1}, y_{1:n-1}\right] &= (1 - \phi_n^2)\Sigma_{n-1}^{s_{i,0}}, \\
\mathrm{Cov}\left[s_{j,n-1}, s_{k,n-1} \middle| s_{i,0}, s_{M,1:n-1}, y_{1:n-1}\right] &= (1 - \phi_n)\Sigma_{n-1}^{s_{i,0},s_{j,0}}, \\
\mathrm{Cov}\left[s_{j,n-1}, s_{M,n} - r_{n,n-1} \middle| s_{i,0}, s_{M,1:n-1}, y_{1:n-1}\right] &= (1 - \phi_n)\Sigma_{n-1}^{s_{i,0},s_{M,n}}.
\end{aligned}$$

Moreover, the second-order condition must be satisfied in every period:

$$\lambda_n - b_n\zeta_n^2 - c_n\delta_n^2 - f_n\zeta_n\delta_n > 0.$$

Finally, conditional (co)variances must satisfy the following recursions:

$$\begin{aligned}
\Sigma_n^{\hat{v}} &= I\theta_I^2\left[\Sigma_n^{s_{i,0}} + (I-1)\Sigma_n^{s_{i,0},s_{j,0}}\right] + 2I(N-n)\theta_I\theta_M\Sigma_n^{s_{i,0},s_{M,n+1}} \\
&\quad + (N-n)\theta_M^2\left[\Sigma_n^{s_{M,n+1}} + (N-n-1)\Sigma_n^{s_{M,n+1},s_{M,n+2}}\right], \\
\Sigma_n^{s_{i,0}} &= \Sigma_{n-1}^{s_{i,0}} - \alpha_n\Sigma_{n-1}^{s_{i,0},s_{M,n}} - \zeta_n\beta_n\left[\Sigma_{n-1}^{s_{i,0}} + (I-1)\Sigma_{n-1}^{s_{i,0},s_{j,0}}\right], \\
\Sigma_n^{s_{i,0},s_{j,0}} &= \Sigma_{n-1}^{s_{i,0},s_{j,0}} - \alpha_n\Sigma_{n-1}^{s_{i,0},s_{M,n}} - \zeta_n\beta_n\left[\Sigma_{n-1}^{s_{i,0}} + (I-1)\Sigma_{n-1}^{s_{i,0},s_{j,0}}\right], \\
\Sigma_n^{s_{i,0},s_{M,n+1}} &= \Sigma_{n-1}^{s_{i,0},s_{M,n}} - \omega_n\Sigma_{n-1}^{s_{i,0},s_{M,n}} - \delta_n\beta_n\left[\Sigma_{n-1}^{s_{i,0}} + (I-1)\Sigma_{n-1}^{s_{i,0},s_{j,0}}\right], \\
\Sigma_n^{s_{M,n+1}} &= \Sigma_{n-1}^{s_{M,n}} - \omega_n\Sigma_{n-1}^{s_{M,n},s_{M,n+1}} - I\delta_n\beta_n\Sigma_{n-1}^{s_{i,0},s_{M,n}}, \\
\Sigma_n^{s_{M,n+1},s_{M,n+2}} &= \Sigma_{n-1}^{s_{M,n},s_{M,n+1}} - \omega_n\Sigma_{n-1}^{s_{M,n},s_{M,n+1}} - I\delta_n\beta_n\Sigma_{n-1}^{s_{i,0},s_{M,n}}.
\end{aligned}$$

Proof. See Appendix 2.A.7.

Similar to Foster and Viswanathan (1996) and Gounas (2021), Proposition 2.1 does not provide analytic results and must be evaluated numerically. Appendix 2.A.8 explains the corresponding backward induction algorithm.

2.5 Numerical Analysis

This section studies the influence of the market maker's short-lived signals on the dynamic trading equilibrium. To this end, let $\rho_0^{\bar{x},\bar{y}}$ be the unconditional correlation coefficient between two arbitrary variables $\bar{x}$ and $\bar{y}$. The main variables of interest are $\rho_0^{s_{i,0},s_{M,n}}$, the unconditional correlation between long-lived and short-lived signals, and $\rho_0^{s_{M,n},s_{M,n+1}}$, the unconditional correlation between future short-lived signals. Note that any given combination of $\rho_0^{s_{i,0},s_{M,n}}$ and $\rho_0^{s_{M,n},s_{M,n+1}}$ is only valid if the covariance matrix in (2.1) is positive definite.

Consider the special case where the combination of long-lived and short-lived information constitutes the truth, that is, $v = \sum_{i=1}^{I} s_{i,0} + \sum_{t=1}^{N} s_{M,t}$. Thus, the true asset value equals the sum of long-lived and short-lived signals. By Lemma 2.1, it follows that $\theta_I = 1$ and $\theta_M = 1$. Additionally, assume that long-lived and short-lived signals have the same initial uncertainty, that is, $\Sigma_0^{s_{i,0}} = \Sigma_0^{s_{M,n}}$ for all informed traders $i \in \{1, \ldots, I\}$ and all periods $n \in \{1, \ldots, N\}$. If Σ_0^{v} is given, then $\Sigma_0^{s_{i,0}}$ and $\Sigma_0^{s_{M,n}}$ immediately follow from (2.16).

The following study considers $I = 3$ informed traders and $N = 10$ trading periods. Moreover, the unconditional asset value variance is standardized to $\Sigma_0^{v} = 1$, and the uninformed order flow variance in each period is set to $\Sigma_0^{u} = 1/N$. For long-lived signals, the unconditional correlations $\rho_0^{s_{i,0},s_{j,0}} \in \{0.25, 0, -0.25\}$ are studied in conjunction with different combinations of $\rho_0^{s_{i,0},s_{M,n}}$ and $\rho_0^{s_{M,n},s_{M,n+1}}$.

2.5.1 Positive Initial Long-Lived Signal Correlation

This analysis sets the initial correlation between long-lived signals to $\rho_0^{s_{i,0},s_{j,0}} = 0.25$ and considers the following initial correlation combinations for short-lived signals: ($\rho_0^{s_{i,0},s_{M,n}} = 0.5$, $\rho_0^{s_{M,n},s_{M,n+1}} = 0.5$), ($\rho_0^{s_{i,0},s_{M,n}} = 0.25$, $\rho_0^{s_{M,n},s_{M,n+1}} = 0.25$), ($\rho_0^{s_{i,0},s_{M,n}} = -0.1$, $\rho_0^{s_{M,n},s_{M,n+1}} = 0.1$), and ($\rho_0^{s_{i,0},s_{M,n}} = 0$, $\rho_0^{s_{M,n},s_{M,n+1}} = -0.1$).

Figure 2.1, Panels A and B plot informed traders' trading intensity β_n and their terminal conditional expected profits over time. First, trading intensities monotonically increase throughout the trading day. Thus, as in Kyle (1985), informed traders submit their most aggressive orders towards the end of the trading day. Second, trading intensities are highest and expected profits are lowest if short-lived signals are initially highly correlated with long-lived and future short-lived signals. In that scenario, the market maker's signals are highly informative about the asset value, resulting in informed traders playing a "rat race" as in Holden and Subrahmanyam (1992).

In contrast, trading intensities are lowest and informed traders make the highest expected profits if short-lived signals are initially independent of long-lived signals and negatively correlated with future short-lived signals. Since short-lived information events are not informative about long-lived signals, informed traders have the highest monopoly power on their long-lived information. As a result, informed traders play a "waiting game" as in Foster and Viswanathan (1996).

Finally, informed traders make approximately the same expected profits in the cases ($\rho_0^{s_{i,0},s_{M,n}} = 0.25$, $\rho_0^{s_{M,n},s_{M,n+1}} = 0.25$) and ($\rho_0^{s_{i,0},s_{M,n}} = -0.1$, $\rho_0^{s_{M,n},s_{M,n+1}} = 0.1$). In the former case, informed traders trade more intensively on their residual information since the market maker's signals are strongly positively correlated with the asset value, generating competition for informed traders. In contrast, the market maker can only gradually learn about the asset value in the latter case, resulting in informed traders submitting less aggressive orders to keep their information private.

Figure 2.1, Panels C and D display the market maker's corresponding news parameter μ_n and illiquidity parameter λ_n over time. It can be inferred that both parameters monotonically decrease throughout the trading day if short-lived signals are initially positively correlated with long-lived and future short-lived signals. In that scenario, news and aggregate order flows are most informative about the asset value at the beginning of the trading day. Consequently, prices are most sensitive in the first trading periods, especially towards news arrivals.

In contrast, if short-lived signals are initially negatively correlated with long-

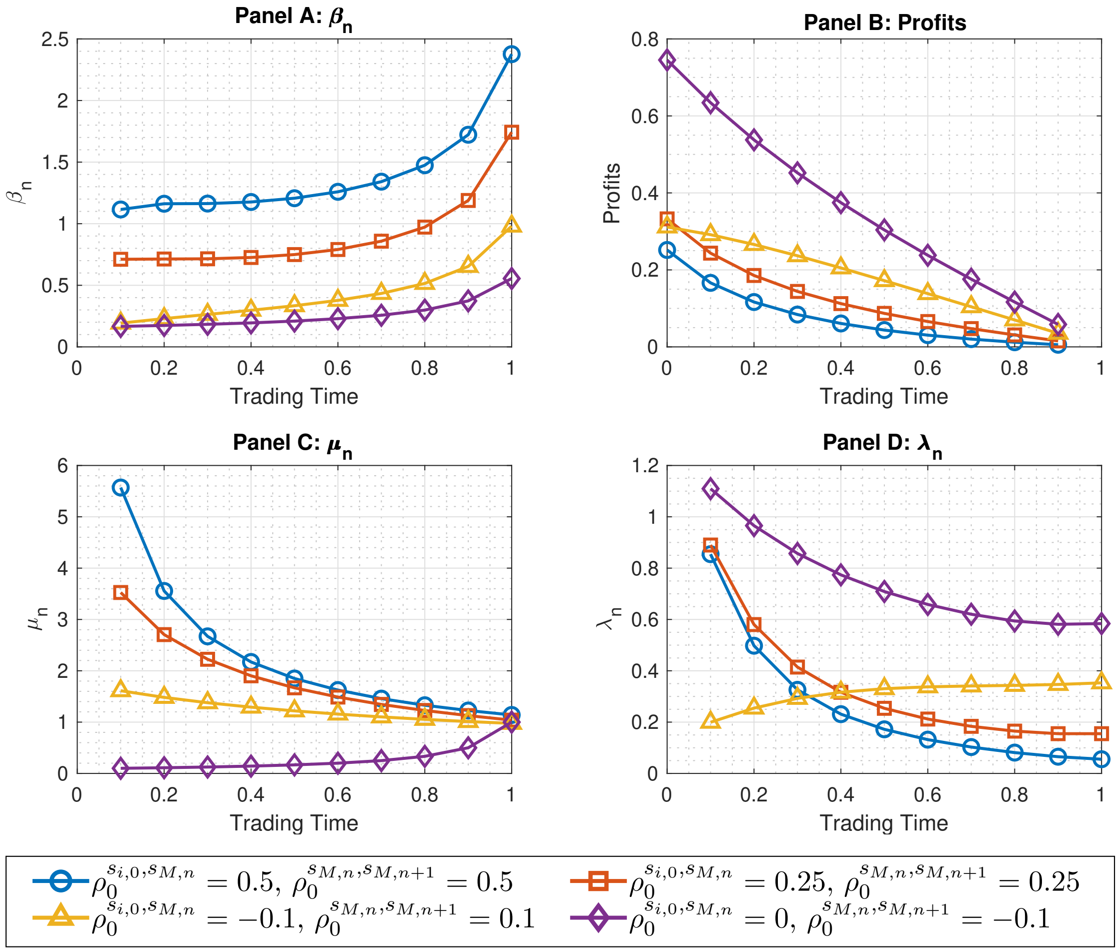

Figure 2.1: Trading Intensity, Expected Profits, News Parameter, and Illiquidity Parameter with Positive Initial Long-Lived Signal Correlation. Informed traders' trading intensity β_n, informed traders' terminal conditional expected profits, the market maker's news parameter μ_n, and the market maker's illiquidity parameter λ_n are plotted over time. The model is solved for the parameter values $I = 3$, $N = 10$, $\Sigma_0^v = 1$, $\Sigma_0^u = 1/N$, $\rho_0^{s_{i,0},s_{j,0}} = 0.25$, and the correlation combinations $(\rho_0^{s_{i,0},s_{M,n}} = 0.5,\ \rho_0^{s_{M,n},s_{M,n+1}} = 0.5)$, $(\rho_0^{s_{i,0},s_{M,n}} = 0.25,\ \rho_0^{s_{M,n},s_{M,n+1}} = 0.25)$, $(\rho_0^{s_{i,0},s_{M,n}} = -0.1,\ \rho_0^{s_{M,n},s_{M,n+1}} = 0.1)$, and $(\rho_0^{s_{i,0},s_{M,n}} = 0,\ \rho_0^{s_{M,n},s_{M,n+1}} = -0.1)$.

lived signals but positively correlated with future short-lived signals, μ_n moderately decreases, while λ_n moderately increases throughout the trading day. In that scenario, short-lived information events and aggregate order flows remain moderately informative about the asset value, which is why the market maker keeps steady price sensitivities over time.

Finally, suppose short-lived signals are initially independent of long-lived signals and negatively correlated with future short-lived signals. In that case, the news parameter is close to zero throughout most of the trading day and only increases in the final periods. Since future short-lived information events are negatively autocorrelated, the market maker effectively smooths her incorporation of news into prices, resulting in the last news innovation having the highest price impact.

Simultaneously, the illiquidity parameter exhibits high values and monotonically decreases over time. Since short-lived information events are independent of long-lived signals, the market maker can only learn through aggregate order flows about informed traders' information. Consequently, prices are highly sensitive to aggregate order flows.

Figure 2.2, Panels A and B plot the conditional asset variance Σ_n^v and the conditional correlation $\rho_n^{s_{i,0},s_{j,0}}$ over time. Naturally, price discovery is highest if the market maker's signals are highly informative about the asset value. In the case ($\rho_0^{s_{i,0},s_{M,n}} = 0.5$, $\rho_0^{s_{M,n},s_{M,n+1}} = 0.5$), prices reflect almost 65% of all available information in the market after the first trading round.

In contrast, price discovery is lowest if short-lived signals are initially independent of long-lived signals and negatively correlated with future short-lived signals. In that case, only aggregate order flows are informative about long-lived signals, and informed traders submit the least aggressive orders. As a result, prices are least revealing about the asset value. Even though less pronounced in later periods, a similar argument applies if short-lived signals are initially negatively correlated with long-lived signals but positively correlated with future short-lived signals.

Finally, as in Foster and Viswanathan (1996), the conditional correlation between long-lived signals becomes negative over time. This result is equivalent to informed traders developing a difference of opinion about the true asset value. Also, it can be inferred that the higher the initial correlation between long-lived and short-lived signals, the more significant the difference of opinion grows.

Figure 2.2, Panels C and D display the evolution of the conditional correlations $\rho_n^{s_{i,0},s_{M,n+1}}$ and $\rho_n^{s_{M,n+1},s_{M,n+2}}$. If short-lived signals are initially positively correlated with long-lived and future short-lived signals, then $\rho_n^{s_{i,0},s_{M,n+1}}$ and $\rho_n^{s_{M,n+1},s_{M,n+2}}$ monotonically decrease over time. Therefore, the market maker's first signal contains the most information about the asset value since subsequent short-lived information events are conditionally less informative.

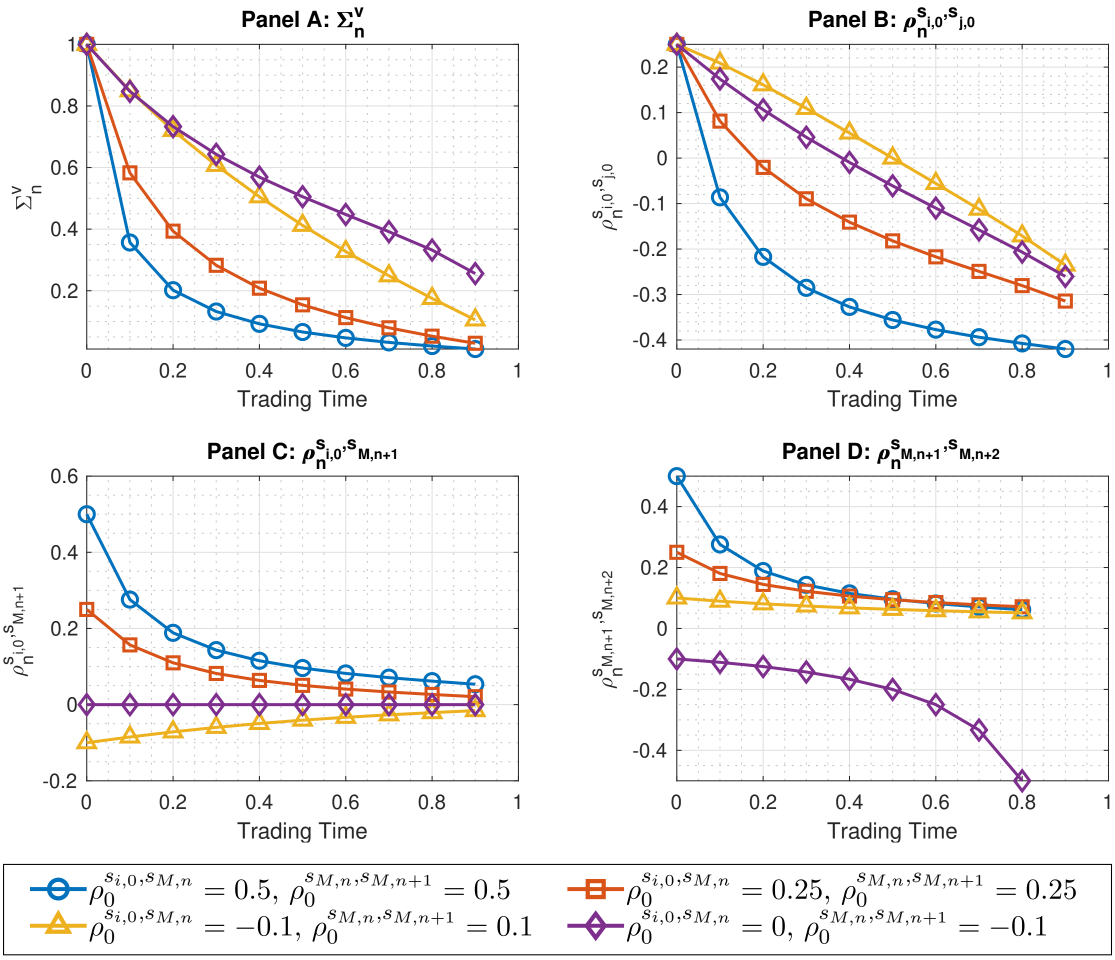

Figure 2.2: Conditional Asset Variance and Conditional Correlations with Positive Initial Long-Lived Signal Correlation. The conditional asset variance Σ_0^v, the conditional correlation between long-lived signals $\rho_n^{s_{i,0}, s_{j,0}}$, the conditional correlation between long-lived and future short-lived signals $\rho_n^{s_{i,0}, s_{M,n+1}}$, and the conditional correlation between future short-lived signals $\rho_n^{s_{M,n+1}, s_{M,n+2}}$ are plotted over time. The model is solved for the parameter values $I = 3$, $N = 10$, $\Sigma_0^v = 1$, $\Sigma_0^u = 1/N$, $\rho_0^{s_{i,0}, s_{j,0}} = 0.25$, and the correlation combinations ($\rho_0^{s_{i,0}, s_{M,n}} = 0.5$, $\rho_0^{s_{M,n}, s_{M,n+1}} = 0.5$), ($\rho_0^{s_{i,0}, s_{M,n}} = 0.25$, $\rho_0^{s_{M,n}, s_{M,n+1}} = 0.25$), ($\rho_0^{s_{i,0}, s_{M,n}} = -0.1$, $\rho_0^{s_{M,n}, s_{M,n+1}} = 0.1$), and ($\rho_0^{s_{i,0}, s_{M,n}} = 0$, $\rho_0^{s_{M,n}, s_{M,n+1}} = -0.1$).

If short-lived signals are initially negatively correlated with long-lived signals but positively correlated with future short-lived signals, $\rho_n^{s_{i,0}, s_{M,n+1}}$ moderately increases, whereas $\rho_n^{s_{M,n+1}, s_{M,n+2}}$ moderately decreases throughout the trading day. Consequently, short-lived information events stay conditionally informative about the asset value, which is why the market maker keeps steady price sensitivities to news innovations and aggregate order flows over time.

Finally, suppose short-lived signals are initially independent of long-lived signals and negatively correlated with future short-lived signals. In that case, the market maker's signals remain conditionally independent of informed traders' information.

However, the conditional correlation between future short-lived signals grows even more negative throughout the trading day. As a result, the market maker smooths the impact of news arrivals on prices.

2.5.2 Zero Initial Long-Lived Signal Correlation

This analysis sets the initial correlation between long-lived signals to $\rho_0^{s_{i,0},s_{j,0}} = 0$ and considers the following initial correlation combinations for short-lived signals: $(\rho_0^{s_{i,0},s_{M,n}} = 0.3,\ \rho_0^{s_{M,n},s_{M,n+1}} = 0.4)$, $(\rho_0^{s_{i,0},s_{M,n}} = 0.2,\ \rho_0^{s_{M,n},s_{M,n+1}} = 0.1)$, $(\rho_0^{s_{i,0},s_{M,n}} = 0,\ \rho_0^{s_{M,n},s_{M,n+1}} = 0)$, and $(\rho_0^{s_{i,0},s_{M,n}} = -0.1,\ \rho_0^{s_{M,n},s_{M,n+1}} = 0.1)$.

Figure 2.3, Panels A and B plot informed traders' trading intensity β_n and their terminal conditional expected profits over time. The higher the initial correlation between the asset value and the market maker's signals, the higher the competition between informed traders, and thus the more significant the trading intensity. Moreover, informed traders have an incentive to delay their trades by submitting their most aggressive orders towards the end of the trading day. However, trading intensities need not be monotonic, as can be inferred from the case $(\rho_0^{s_{i,0},s_{M,n}} = 0.3,\ \rho_0^{s_{M,n},s_{M,n+1}} = 0.4)$.

Naturally, informed traders' expected profits are highest if either the market maker's signals are initially moderately informative or not informative at all about long-lived and future short-lived signals. Intuitively, if short-lived information events are not too informative about the asset value, informed traders enjoy some degree of monopoly power and make high expected profits as a result.

In contrast, informed traders make the lowest expected profits in the cases $(\rho_0^{s_{i,0},s_{M,n}} = 0.3,\ \rho_0^{s_{M,n},s_{M,n+1}} = 0.4)$ and $(\rho_0^{s_{i,0},s_{M,n}} = -0.1,\ \rho_0^{s_{M,n},s_{M,n+1}} = 0.1)$. In the former case, competition deteriorates profits in later periods so that informed traders realize most profits at the beginning of the trading day, while in the latter case, informed traders make their largest profits in the final periods when they submit their most aggressive orders.

Figure 2.3, Panels C and D display the market maker's corresponding news

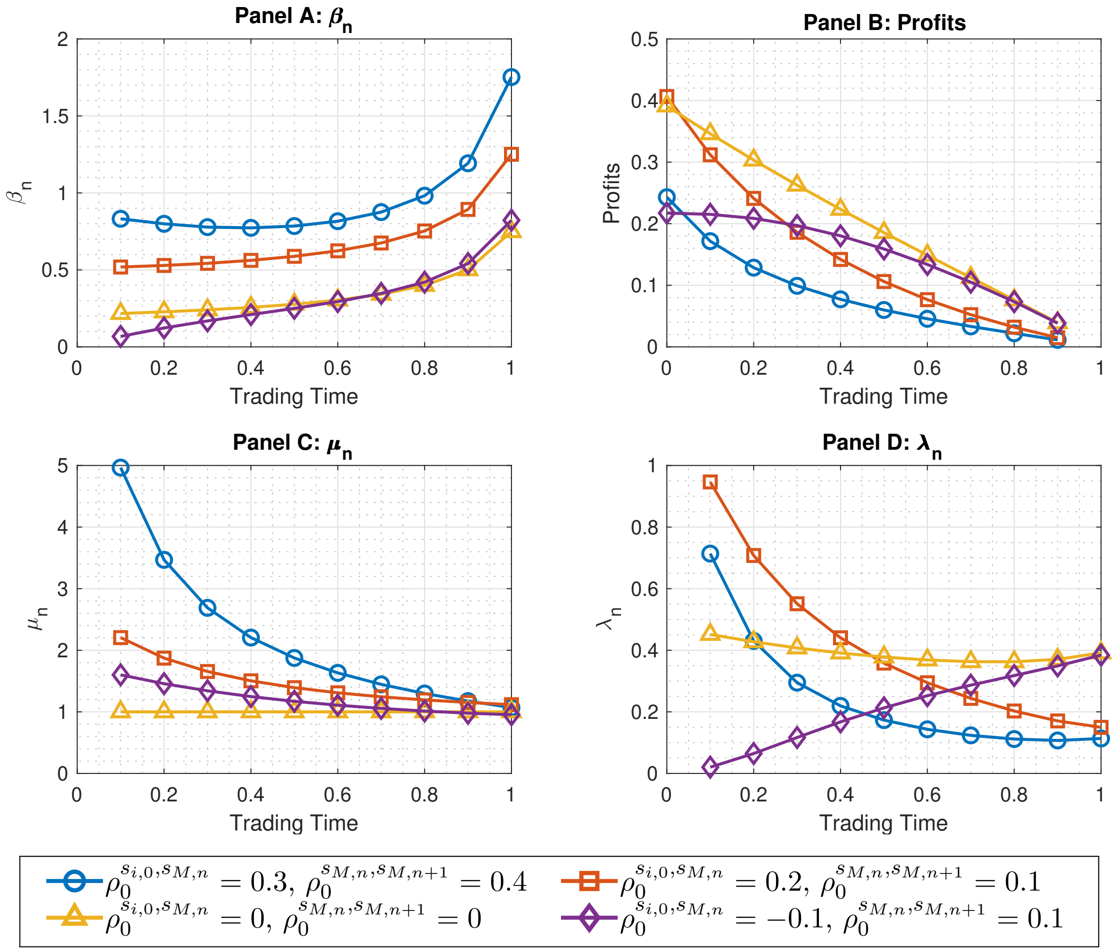

Figure 2.3: Trading Intensity, Expected Profits, News Parameter, and Illiquidity Parameter with Zero Initial Long-Lived Signal Correlation. Informed traders' trading intensity β_n, informed traders' terminal conditional expected profits, the market maker's news parameter μ_n, and the market maker's illiquidity parameter λ_n are plotted over time. The model is solved for the parameter values $I = 3$, $N = 10$, $\Sigma_0^v = 1$, $\Sigma_0^u = 1/N$, $\rho_0^{s_{i,0},s_{j,0}} = 0$, and the correlation combinations ($\rho_0^{s_{i,0},s_{M,n}} = 0.3$, $\rho_0^{s_{M,n},s_{M,n+1}} = 0.4$), ($\rho_0^{s_{i,0},s_{M,n}} = 0.2$, $\rho_0^{s_{M,n},s_{M,n+1}} = 0.1$), ($\rho_0^{s_{i,0},s_{M,n}} = 0$, $\rho_0^{s_{M,n},s_{M,n+1}} = 0$), and ($\rho_0^{s_{i,0},s_{M,n}} = -0.1$, $\rho_0^{s_{M,n},s_{M,n+1}} = 0.1$).

parameter μ_n and illiquidity parameter λ_n over time. If short-lived signals are initially positively correlated with long-lived and future short-lived signals, then μ_n and λ_n monotonically decrease throughout the trading day. Thus, news and aggregate order flows contain the most information about the asset value in the first periods. In particular, prices are highly sensitive to news arrivals at the beginning of the trading day if news arrivals are highly informative about the asset value.

If short-lived signals are initially independent of long-lived and future short-lived signals, the news parameter μ_n is equal to one in every period. Since news arrives independently over time, each news has the same price impact. Even though visually not distinguishable due to scale, the illiquidity parameter λ_n exhibits a U-shape. It initially declines because aggregate order flows are most informative about the asset value at the beginning of the trading day. However, the illiquidity parameter increases in the final periods when informed traders trade most aggressively on their residual information.

Finally, suppose short-lived signals are initially negatively correlated with long-lived signals but positively correlated with future short-lived signals. In that case, μ_n monotonically decreases, whereas λ_n monotonically increases over time. Since the market maker's signals are informative about future short-lived signals, the first news arrival has a price impact larger than one. In contrast, the market maker increases her illiquidity parameter throughout the trading day since she can extract the least information from aggregate order flows in the first periods.

Figure 2.4, Panels A and B plot the conditional asset variance Σ_n^v and the conditional correlation $\rho_n^{s_{i,0},s_{j,0}}$ over time. The higher the initial correlation between the asset value and the market maker's signals, the faster information about the asset value gets incorporated into prices. In particular, price discovery is lowest if short-lived signals are independently distributed of long-lived and future short-lived signals. Nonetheless, prices still reflect almost 85% of all available information in the market at the end of the trading day.

Moreover, the conditional correlation between long-lived signals immediately grows negative. As in Foster and Viswanathan (1996), informed traders with long-lived independent signals develop a difference of opinion about the true asset value after only one trading period. Also, the difference of opinion grows faster with a higher initial correlation between long-lived and short-lived signals.

Figure 2.4, Panels C and D display the evolution of the conditional correlations $\rho_n^{s_{i,0},s_{M,n+1}}$ and $\rho_n^{s_{M,n+1},s_{M,n+2}}$. If short-lived signals are initially positively correlated with long-lived and future short-lived signals, then $\rho_n^{s_{i,0},s_{M,n+1}}$ and $\rho_n^{s_{M,n+1},s_{M,n+2}}$ monotonically decrease throughout the trading day. Since news arrivals are most informative about the asset value in the first periods, price discovery is high.

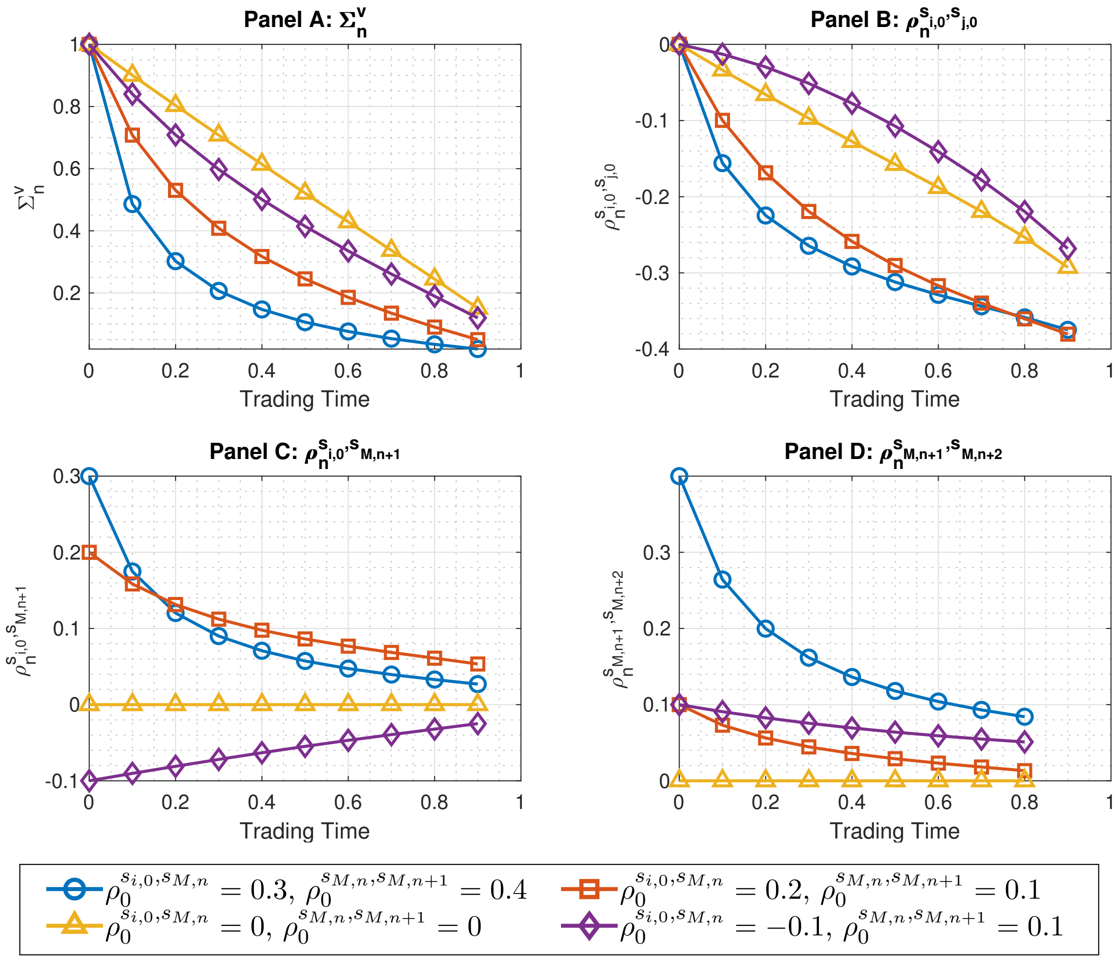

Figure 2.4: Conditional Asset Variance and Conditional Correlations with Zero Initial Long-Lived Signal Correlation. The conditional asset variance Σ_0^v, the conditional correlation between long-lived signals $\rho_n^{s_{i,0},s_{j,0}}$, the conditional correlation between long-lived and future short-lived signals $\rho_n^{s_{i,0},s_{M,n+1}}$, and the conditional correlation between future short-lived signals $\rho_n^{s_{M,n+1},s_{M,n+2}}$ are plotted over time. The model is solved for the parameter values $I = 3$, $N = 10$, $\Sigma_0^v = 1$, $\Sigma_0^u = 1/N$, $\rho_0^{s_{i,0},s_{j,0}} = 0$, and the correlation combinations ($\rho_0^{s_{i,0},s_{M,n}} = 0.3$, $\rho_0^{s_{M,n},s_{M,n+1}} = 0.4$), ($\rho_0^{s_{i,0},s_{M,n}} = 0.2$, $\rho_0^{s_{M,n},s_{M,n+1}} = 0.1$), ($\rho_0^{s_{i,0},s_{M,n}} = 0$, $\rho_0^{s_{M,n},s_{M,n+1}} = 0$), and ($\rho_0^{s_{i,0},s_{M,n}} = -0.1$, $\rho_0^{s_{M,n},s_{M,n+1}} = 0.1$).

If short-lived signals are initially independent of long-lived and future short-lived signals, they also remain independent upon conditioning. In that scenario, news arrives independently over time, not containing any information about informed traders' signals or future news. As a result, price discovery is low in comparison.

Finally, suppose short-lived signals are initially negatively correlated with long-lived signals but positively correlated with future short-lived signals. In that case, $\rho_n^{s_{i,0},s_{M,n+1}}$ monotonically increases, whereas $\rho_n^{s_{M,n+1},s_{M,n+2}}$ monotonically decreases over time. Therefore, the market maker's signals become conditionally less informative about informed traders' signals and future news arrivals in later periods.

2.5.3 Negative Initial Long-Lived Signal Correlation

This analysis sets the initial correlation between long-lived signals to $\rho_0^{s_{i,0},s_{j,0}} = -0.25$ and considers the following initial correlation combinations for short-lived signals: $(\rho_0^{s_{i,0},s_{M,n}} = 0.35,\ \rho_0^{s_{M,n},s_{M,n+1}} = 0.75)$, $(\rho_0^{s_{i,0},s_{M,n}} = 0.25,\ \rho_0^{s_{M,n},s_{M,n+1}} = 0.35)$, $(\rho_0^{s_{i,0},s_{M,n}} = 0.1,\ \rho_0^{s_{M,n},s_{M,n+1}} = 0)$, and $(\rho_0^{s_{i,0},s_{M,n}} = -0.1,\ \rho_0^{s_{M,n},s_{M,n+1}} = 0.1)$.

Figure 2.5, Panels A and B plot informed traders' trading intensity β_n and their terminal conditional expected profits over time. Similar to the previous analysis, informed traders submit their most aggressive orders towards the end of the trading day. Moreover, it can be inferred that a high initial correlation between the asset value and short-lived signals implies large trading intensities and low expected profits for informed traders.

The case $(\rho_0^{s_{i,0},s_{M,n}} = -0.1,\ \rho_0^{s_{M,n},s_{M,n+1}} = 0.1)$ is of particular interest. Surprisingly, trading intensities are negative in the first four periods. Thus, informed traders initially trade against their private information. Recall that the asset value is assumed to be the sum of long-lived and short-lived signals. Since each informed trader's signal is initially negatively correlated with other informed traders' signals and the market maker's signals, the initial correlation between each informed trader's signal and the asset value is also negative. Consequently, it is optimal for informed traders to trade against their signals at the beginning of the trading day. However, as informed traders sequentially learn about the market maker's signals, the conditional correlation between each informed trader's signal and the asset value eventually becomes positive. As a result, informed traders change trading directions in later periods.

Figure 2.5, Panels C and D display the market maker's corresponding news parameter μ_n and illiquidity parameter λ_n over time. If short-lived signals are initially positively correlated with long-lived and future short-lived signals, then μ_n and λ_n monotonically decrease throughout the trading day. Moreover, the magnitude of the parameters depends on the correlation between the asset value and the market maker's signals. Prices are especially sensitive to news arrivals if

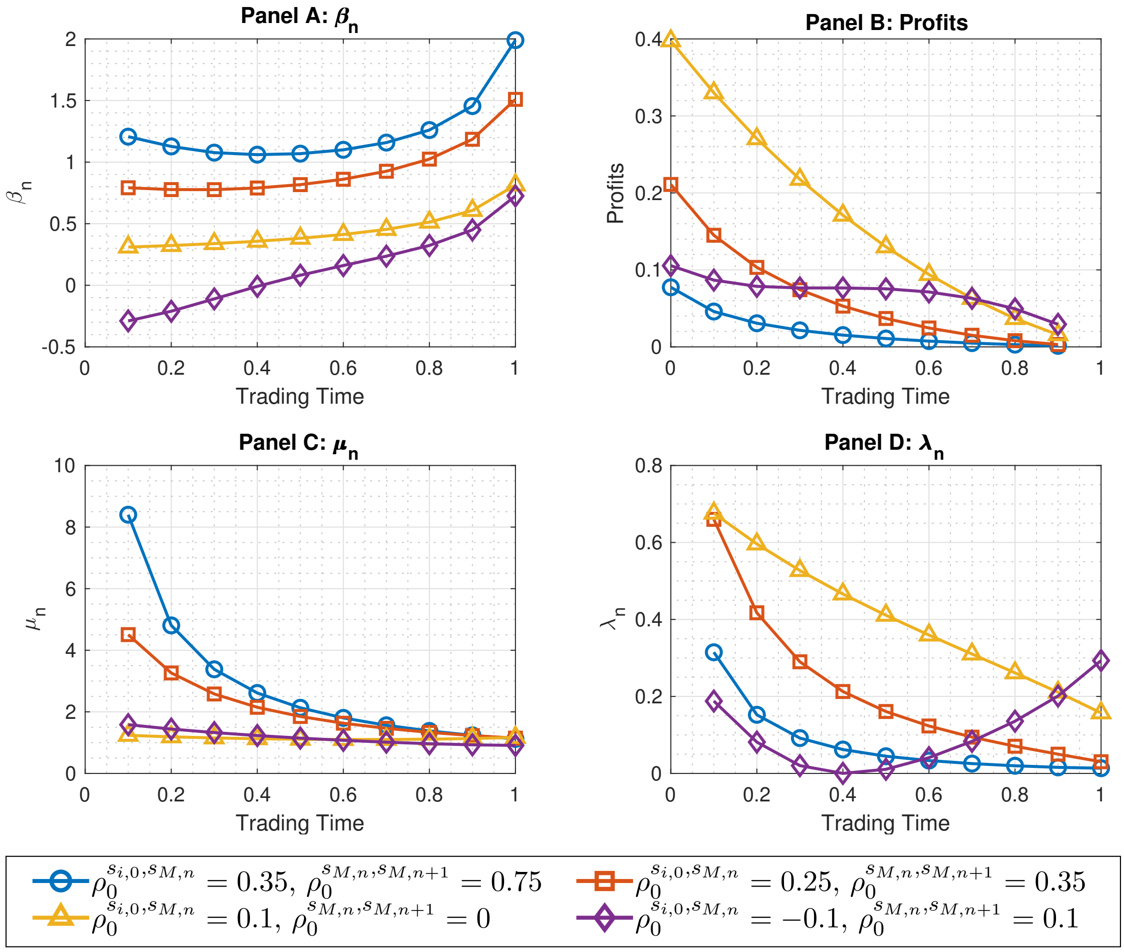

Figure 2.5: Trading Intensity, Expected Profits, News Parameter, and Illiquidity Parameter with Negative Initial Long-Lived Signal Correlation. Informed traders' trading intensity β_n, informed traders' terminal conditional expected profits, the market maker's news parameter μ_n, and the market maker's illiquidity parameter λ_n are plotted over time. The model is solved for the parameter values $I = 3$, $N = 10$, $\Sigma_0^v = 1$, $\Sigma_0^u = 1/N$, $\rho_0^{s_{i,0},s_{j,0}} = -0.25$, and the correlation combinations ($\rho_0^{s_{i,0},s_{M,n}} = 0.35$, $\rho_0^{s_{M,n},s_{M,n+1}} = 0.75$), ($\rho_0^{s_{i,0},s_{M,n}} = 0.25$, $\rho_0^{s_{M,n},s_{M,n+1}} = 0.35$), ($\rho_0^{s_{i,0},s_{M,n}} = 0.1$, $\rho_0^{s_{M,n},s_{M,n+1}} = 0$), and ($\rho_0^{s_{i,0},s_{M,n}} = -0.1$, $\rho_0^{s_{M,n},s_{M,n+1}} = 0.1$).

short-lived information events are highly informative about the asset value.

Suppose short-lived signals are initially positively correlated with long-lived signals and independent of future short-lived signals. In that case, the illiquidity parameter is high in magnitude and only gradually decreases throughout the trading day. Since short-lived signals help the market maker extract information about long-lived signals from aggregate order flows, the market maker keeps a steady order flow sensitivity as a result.

Even though visually not distinguishable due to scale, the news parameter is U-shaped in that scenario. Therefore, prices are most sensitive to news innovations at the beginning and end of the trading day. Similar to the previous results, μ_n initially declines because the first news arrival contains the most information about the asset value. However, since the correlation between consecutive future short-lived information events grows negative throughout the trading day, the market maker eventually smooths her incorporation of news into prices, resulting in μ_n increasing in the final periods. To conclude, the news parameter does not need to be monotonic.

Finally, suppose short-lived signals are initially negatively correlated with long-lived signals but positively correlated with future short-lived signals. In that case, μ_n monotonically decreases over time, while λ_n exhibits a clear U-shape. Note that the evolution of λ_n is directly linked to informed traders' trading intensities. The illiquidity parameter initially declines because trading intensities decrease in absolute value in the first periods. In the fourth trading round, λ_n is approximately zero and the market is most liquid because informed traders do not submit orders in this period. Finally, the market maker increases her price sensitivity to aggregate order flows in the final trading rounds when informed traders submit their most aggressive orders.

Figure 2.6, Panels A and B plot the conditional asset variance Σ_n^v and the conditional correlation $\rho_n^{s_{i,0},s_{j,0}}$ over time. The higher the initial correlation between the asset value and the market maker's signals, the higher the price discovery throughout the trading day. In the case ($\rho_0^{s_{i,0},s_{M,n}} = 0.35$, $\rho_0^{s_{M,n},s_{M,n+1}} = 0.75$), prices reveal almost 78.5% of the asset value after the first trading round. Moreover, note that price discovery is lowest and approximately the same in the cases ($\rho_0^{s_{i,0},s_{M,n}} = 0.1$, $\rho_0^{s_{M,n},s_{M,n+1}} = 0$) and ($\rho_0^{s_{i,0},s_{M,n}} = -0.1$, $\rho_0^{s_{M,n},s_{M,n+1}} = 0.1$), even though informed traders' expected profits are almost four times higher in the former case.

Similar to the previous findings, the conditional correlation between long-lived signals monotonically decreases over time. Moreover, it can be inferred that the initial correlation between long-lived and short-lived signals determines the speed at which informed traders' difference of opinion about the true asset value grows.

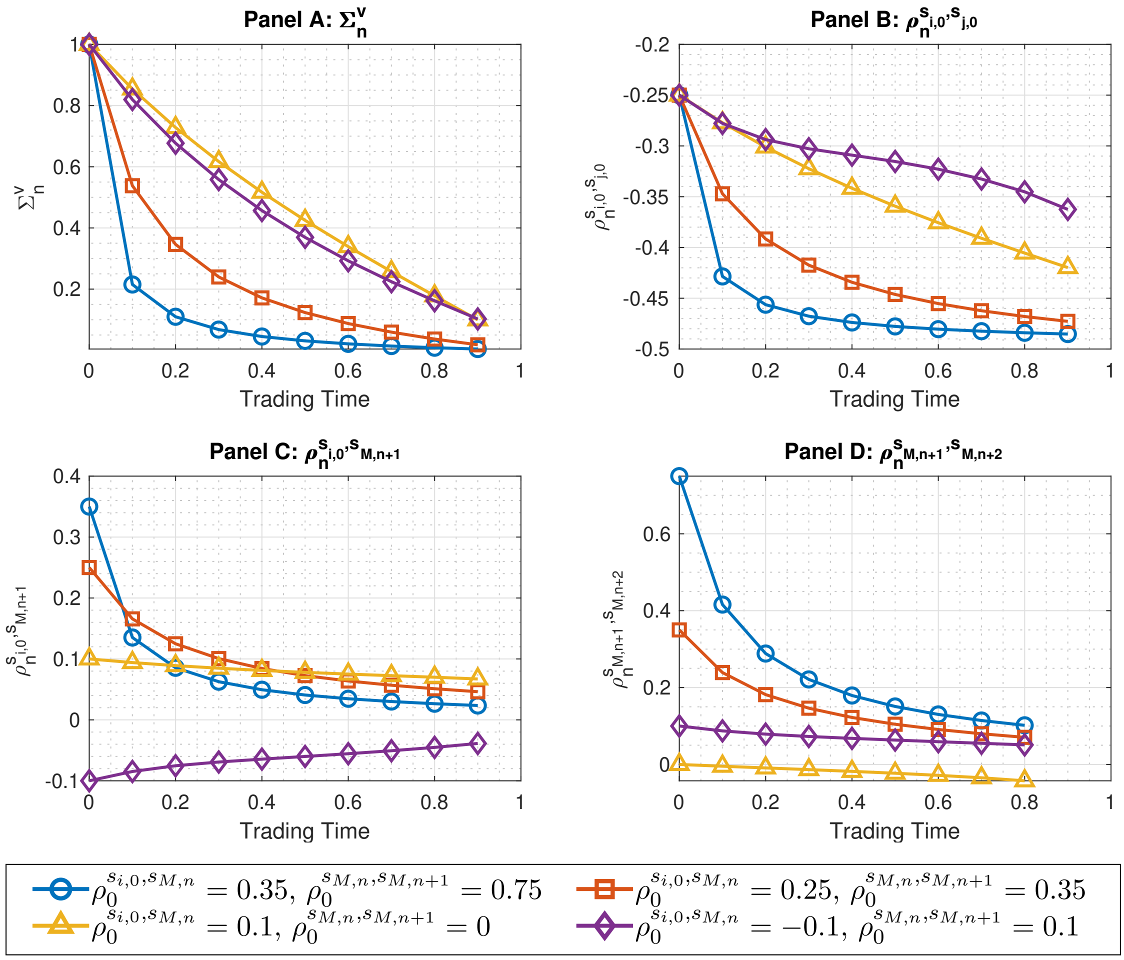

Figure 2.6: Conditional Asset Variance and Conditional Correlations with Negative Initial Long-Lived Signal Correlation. The conditional asset variance Σ_0^v, the conditional correlation between long-lived signals $\rho_n^{s_{i,0},s_{j,0}}$, the conditional correlation between long-lived and future short-lived signals $\rho_n^{s_{i,0},s_{M,n+1}}$, and the conditional correlation between future short-lived signals $\rho_n^{s_{M,n+1},s_{M,n+2}}$ are plotted over time. The model is solved for the parameter values $I = 3$, $N = 10$, $\Sigma_0^v = 1$, $\Sigma_0^u = 1/N$, $\rho_0^{s_{i,0},s_{j,0}} = -0.25$, and the correlation combinations ($\rho_0^{s_{i,0},s_{M,n}} = 0.35$, $\rho_0^{s_{M,n},s_{M,n+1}} = 0.75$), ($\rho_0^{s_{i,0},s_{M,n}} = 0.25$, $\rho_0^{s_{M,n},s_{M,n+1}} = 0.35$), ($\rho_0^{s_{i,0},s_{M,n}} = 0.1$, $\rho_0^{s_{M,n},s_{M,n+1}} = 0$), and ($\rho_0^{s_{i,0},s_{M,n}} = -0.1$, $\rho_0^{s_{M,n},s_{M,n+1}} = 0.1$).

Figure 2.6, Panels C and D display the evolution of the conditional correlations $\rho_n^{s_{i,0},s_{M,n+1}}$ and $\rho_n^{s_{M,n+1},s_{M,n+2}}$. It can be inferred that both conditional correlations monotonically decrease over time if short-lived signals are initially positively correlated with long-lived and future short-lived signals. In that scenario, short-lived information events are most informative about the asset value at the beginning of the trading day. Consequently, prices become less sensitive to news innovations and aggregate order flows in later periods.

If short-lived signals are initially positively correlated with long-lived signals and independent of future short-lived signals, then $\rho_n^{s_{i,0},s_{M,n+1}}$ and $\rho_n^{s_{M,n+1},s_{M,n+2}}$ only slowly decrease over time. In that scenario, short-lived information events remain consistently informative about the asset value throughout the trading day. However, the conditional correlation between the asset value and short-lived signals remains small in magnitude, resulting in low price discovery in comparison.

Finally, suppose short-lived signals are initially negatively correlated with long-lived signals but positively correlated with future short-lived signals. In that case, $\rho_n^{s_{i,0},s_{M,n+1}}$ moderately increases, whereas $\rho_n^{s_{M,n+1},s_{M,n+2}}$ slowly decreases over time. Thus, short-lived information events remain consistently autocorrelated and become conditionally less informative about informed traders' signals throughout the trading day.

2.6 Conclusion

This paper has studied a strategic trading model in which the market maker has a monopoly on short-lived information. Necessary and sufficient conditions for a dynamic trading equilibrium were provided, and a numerical study was performed. Interpreting short-lived information events as news, the correlation structure of news arrivals has significant effects on trading strategies, market liquidity, and price discovery. In particular, the model can generate new stylized facts like negative trading intensities and increasing price sensitivities to news.

One shortcoming of this paper's model, shared by most strategic trading models, is that aggregate order flows are unpredictable. In contrast, empirical order flows exhibit significant autocorrelation. Therefore, one interesting future research is to combine this paper's framework with the model in Gounas (2021), which explicitly accounts for autocorrelated order flows. Moreover, the derivation of the dynamic trading equilibrium relies on symmetry conditions for (co)variances, and it remains an open topic whether this assumption can be relaxed without affecting tractability in a discrete-time setting. Finally, this paper assumes that short-lived information events arrive deterministically over time, whereas news arrivals in financial markets can also be stochastic. Consequently, it would be interesting to extend the model by allowing short-lived information events to arrive stochastically throughout the trading day.

2.A Appendix

2.A.1 Proof of Lemma 2.1

Proof. Assume the distribution assumption (2.1) holds. Then the projection theorem for Gaussian random variables implies

$$\mathbb{E}\left[v\middle|\boldsymbol{s_I},\boldsymbol{s_M}\right]=\begin{pmatrix}\Sigma_0^{v,\boldsymbol{s_I}} & \Sigma_0^{v,\boldsymbol{s_M}}\end{pmatrix}\begin{pmatrix}\Sigma_0^{\boldsymbol{s_I}} & \Sigma_0^{\boldsymbol{s_I},\boldsymbol{s_M}}\\ \Sigma_0^{\boldsymbol{s_M},\boldsymbol{s_I}} & \Sigma_0^{\boldsymbol{s_M}}\end{pmatrix}^{-1}\begin{pmatrix}\boldsymbol{s_I}\\ \boldsymbol{s_M}\end{pmatrix}. \tag{2.A1}$$

Since (co)variances are assumed to be symmetrical, it follows that

$$\mathbb{E}\left[v\middle|\boldsymbol{s_I},\boldsymbol{s_M}\right]=\begin{pmatrix}\theta_I\boldsymbol{\iota_I} & \theta_M\boldsymbol{\iota_N}\end{pmatrix}\begin{pmatrix}\boldsymbol{s_I}\\ \boldsymbol{s_M}\end{pmatrix}, \tag{2.A2}$$

where θ_I and θ_M are constants and $\boldsymbol{\iota_I}$ and $\boldsymbol{\iota_N}$ are $(1\times I)$ and $(1\times N)$ vectors of ones, respectively. Consequently,

$$\mathbb{E}\left[v\middle|\boldsymbol{s_I},\boldsymbol{s_M}\right]=\theta_I\sum_{i=1}^{I}s_{i,0}+\theta_M\sum_{t=1}^{N}s_{M,t}\equiv\hat{v}. \tag{2.A3}$$

Since $\hat{v}$ captures all available information in the market, it is a sufficient statistic to forecast the asset value v. □

2.A.2 Proof of Lemma 2.2

Proof. Given (2.7), applying the projection theorem for Gaussian random variables yields for all $n \in \{1, \ldots, N\}$:

$$\begin{aligned} p_n &= \mathbb{E}\left[v \middle| s_{M,1:n}, y_{1:n}\right] \\ &= p_{n-1} + \mathbb{E}\left[v - p_{n-1} \middle| s_{M,1:n}, y_{1:n}\right] \\ &= p_{n-1} + \mathbb{E}\left[v - p_{n-1} \middle| s_{M,1:n-1}, s_{M,n} - r_{n,n-1}, y_{1:n-1}, y_n\right] \\ &= p_{n-1} + \mathbb{E}\left[v - p_{n-1} \middle| s_{M,n} - r_{n,n-1}, y_n\right] \\ &= p_{n-1} + \mu_n(s_{M,n} - r_{n,n-1}) + \lambda_n y_n. \end{aligned} \tag{2.A4}$$

Moreover, the symmetry conditions for (co)variances imply for all $i \in \{1, \ldots, I\}$ and all $n \in \{1, \ldots, N\}$:

$$\begin{aligned} t_{i,n} &= \mathbb{E}\left[s_{i,0} \middle| s_{M,1:n}, y_{1:n}\right] \\ &= t_{i,n-1} + \mathbb{E}\left[s_{i,0} - t_{i,n-1} \middle| s_{M,1:n-1}, s_{M,n} - r_{n,n-1}, y_{1:n-1}, y_n\right] \\ &= t_{i,n-1} + \mathbb{E}\left[s_{i,n-1} \middle| s_{M,n} - r_{n,n-1}, y_n\right] \\ &= t_{i,n-1} + \alpha_n(s_{M,n} - r_{n,n-1}) + \zeta_n y_n. \end{aligned} \tag{2.A5}$$

A similar analysis yields for all $n \in \{1, \ldots, N-1\}$ and $t \in \{1, \ldots, N-n\}$:

$$\begin{aligned} r_{n+t,n} &= \mathbb{E}\left[s_{M,n+t} \middle| s_{M,1:n}, y_{1:n}\right] \\ &= r_{n+t,n-1} + \mathbb{E}\left[s_{M,n+t} - r_{n+t,n-1} \middle| s_{M,1:n-1}, s_{M,n} - r_{n,n-1}, y_{1:n-1}, y_n\right] \\ &= r_{n+t,n-1} + \mathbb{E}\left[s_{M,n+t} - r_{n+t,n-1} \middle| s_{M,n} - r_{n,n-1}, y_n\right] \\ &= r_{n,n-1} + \mathbb{E}\left[s_{M,n+1} - r_{n,n-1} \middle| s_{M,n} - r_{n,n-1}, y_n\right] \\ &= r_{n,n-1} + \omega_n(s_{M,n} - r_{n,n-1}) + \delta_n y_n. \end{aligned} \tag{2.A6}$$

Finally, to show the relations for μ_n and λ_n, the law of iterated expectations implies

$$\begin{aligned}
p_n &= \mathbb{E}\left[v \middle| s_{M,1:n}, y_{1:n}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[v \middle| \boldsymbol{s_I}, \boldsymbol{s_M}\right] \middle| s_{M,1:n}, y_{1:n}\right] \\
&= \mathbb{E}\left[\hat{v} \middle| s_{M,1:n}, y_{1:n}\right] \\
&= \mathbb{E}\left[\theta_I \sum_{i=1}^{I} s_{i,0} + \theta_M \sum_{t=1}^{N} s_{M,t} \middle| s_{M,1:n}, y_{1:n}\right] \\
&= \theta_I \sum_{i=1}^{I} t_{i,n} + \theta_M \left[\sum_{t=1}^{n} s_{M,t} + \sum_{t=n+1}^{N} r_{t,n}\right] \\
&= I\theta_I t_{i,n} + \theta_M \left[\sum_{t=1}^{n} s_{M,t} + (N-n) r_{n+1,n}\right].
\end{aligned} \tag{2.A7}$$

Consequently,

$$\begin{aligned}
p_n - p_{n-1} &= I\theta_I(t_{i,n} - t_{i,n-1}) + \theta_M\Big[s_{M,n} - r_{n,n-1} + (N-n)(r_{n+1,n} - r_{n,n-1})\Big] \\
&= \Big(I\theta_I\alpha_n + \theta_M\Big[1 + (N-n)\omega_n\Big]\Big)(s_{M,n} - r_{n,n-1}) + \Big[I\theta_I\zeta_n + (N-n)\theta_M\delta_n\Big]y_n \\
&= \mu_n(s_{M,n} - r_{n,n-1}) + \lambda_n y_n,
\end{aligned} \tag{2.A8}$$

as was to be shown. □

2.A.3 Proof of Lemma 2.3

Proof. The first part of the lemma is shown in Foster and Viswanathan (1996) and Gounas (2021), while the second part of the lemma is derived as follows:

$$
\begin{aligned}
\Sigma_n^{\hat{v}} &= \mathrm{Var}\left[\hat{v}\middle|s_{M,1:n}, y_{1:n}\right] \\
&= \mathrm{Var}\left[\theta_I \sum_{i=1}^{I} s_{i,0} + \theta_M \sum_{t=1}^{N} s_{M,t}\middle|s_{M,1:n}, y_{1:n}\right] \\
&= \mathbb{E}\left[\left(\theta_I \sum_{i=1}^{I} s_{i,0} + \theta_M \sum_{t=1}^{N} s_{M,t} - \theta_I \sum_{i=1}^{I} t_{i,n} - \theta_M \left[\sum_{t=1}^{n} s_{M,t} + \sum_{t=n+1}^{N} r_{t,n}\right]\right)^2\right] \\
&= \mathbb{E}\left[\left(\theta_I \sum_{i=1}^{I} s_{i,n} + \theta_M \sum_{t=n+1}^{N} \left[s_{M,t} - r_{t,n}\right]\right)^2\right] \\
&= I\theta_I^2\left[\Sigma_n^{s_{i,0}} + (I-1)\Sigma_n^{s_{i,0},s_{j,0}}\right] + 2I(N-n)\theta_I\theta_M\Sigma_n^{s_{i,0},s_{M,n+1}} \\
&\quad + (N-n)\theta_M^2\left[\Sigma_n^{s_{M,n+1}} + (N-n-1)\Sigma_n^{s_{M,n+1},s_{M,n+2}}\right],
\end{aligned}
\tag{2.A9}
$$

as was to be shown. □

2.A.4 Proof of Lemma 2.4

Proof. Given (2.7), (2.8), (2.9), and (2.10), applying the projection theorem for Gaussian random variables yields for all $n \in \{1, \ldots, N\}$ and $i \in \{1, \ldots, I\}$:

$$
\begin{aligned}
\mathbb{E}\left[s_{M,n}\middle|s_{i,0}, s_{M,1:n-1}, y_{1:n-1}, x_{i,1:n-1}\right] &= \mathbb{E}\left[s_{M,n}\middle|s_{i,0}, s_{M,1:n-1}, y_{1:n-1}\right] \\
&= r_{n,n-1} + \mathbb{E}\left[s_{M,n} - r_{n,n-1}\middle|s_{i,0}, s_{M,1:n-1}, y_{1:n-1}\right] \\
&= r_{n,n-1} + \mathbb{E}\left[s_{M,n} - r_{n,n-1}\middle|s_{i,n-1}, s_{M,1:n-1}, y_{1:n-1}\right] \\
&= r_{n,n-1} + \mathbb{E}\left[s_{M,n} - r_{n,n-1}\middle|s_{i,n-1}\right] \\
&= r_{n,n-1} + \psi_n s_{i,n-1}.
\end{aligned}
\tag{2.A10}
$$

A similar analysis shows that

$$
\begin{aligned}
\mathbb{E}\left[v\middle|s_{i,0}, s_{M,1:n-1}, y_{1:n-1}, x_{i,1:n-1}\right] &= p_{n-1} + \mathbb{E}\left[v - p_{n-1}\middle|s_{i,n-1}\right] \\
&= p_{n-1} + \eta_n s_{i,n-1},
\end{aligned}
\tag{2.A11}
$$

and

$$\mathbb{E}\left[s_{j,0}\middle|s_{i,0},s_{M,1:n-1},y_{1:n-1},x_{i,1:n-1}\right] = t_{j,n-1} + \mathbb{E}\left[s_{j,n-1}\middle|s_{i,n-1}\right] \\ = t_{j,n-1} + \phi_n s_{i,n-1}, \tag{2.A12}$$

where $j \in \{1,\ldots,I\}$ and $j \neq i$. □

2.A.5 Proof of Lemma 2.5

Proof. See Foster and Viswanathan (1996) and Gounas (2021). □

2.A.6 Proof of Lemma 2.6

Proof. Given (2.7), (2.8), (2.9), and (2.10), assume any informed trader $i \in \{1,\ldots,I\}$ has deviated from (2.7). Then Lemma 2.5 implies for all $n \in \{1,\ldots,N\}$:

$$\begin{aligned} &\mathbb{E}\left[s_{M,n} - r_{n,n-1}\middle|s_{i,0},s_{M,1:n-1},y_{1:n-1},\tilde{x}_{i,1:n-1}\right] \\ =&\mathbb{E}\left[s_{M,n} - \hat{r}^i_{n,n-1} + \hat{r}^i_{n,n-1} - r_{n,n-1}\middle|s_{i,0},s_{M,1:n-1},y_{1:n-1},\tilde{x}_{i,1:n-1}\right] \\ =&\mathbb{E}\left[s_{M,n} - \hat{r}^i_{n,n-1}\middle|s_{i,0},s_{M,1:n-1},y_{n-1},\tilde{x}_{i,1:n-1}\right] + \left(\hat{r}^i_{n,n-1} - r_{n,n-1}\right). \end{aligned} \tag{2.A13}$$

Applying the projection theorem for Gaussian random variables yields for the term on the left-hand side:[10]

$$\begin{aligned} &\mathbb{E}\left[s_{M,n} - \hat{r}^i_{n,n-1}\middle|s_{i,0},s_{M,1:n-1},y_{1:n-1},\tilde{x}_{i,1:n-1}\right] \\ =&\mathbb{E}\left[s_{M,n} - \hat{r}^i_{n,n-1}\middle|s_{i,0},s_{M,1:n-1},\hat{y}^i_{1:n-1},\tilde{x}_{i,1:n-1}\right] \\ =&\mathbb{E}\left[s_{M,n} - \hat{r}^i_{n,n-1}\middle|\hat{s}^i_{i,n-1},s_{M,1:n-1},\hat{y}^i_{1:n-1},\tilde{x}_{i,1:n-1}\right] \\ =&\mathbb{E}\left[s_{M,n} - \hat{r}^i_{n,n-1}\middle|\hat{s}^i_{i,n-1},\tilde{x}_{i,1:n-1}\right] \\ =&\mathbb{E}\left[s_{M,n} - \hat{r}^i_{n,n-1}\middle|\hat{s}^i_{i,n-1}\right] \\ =&\psi_n \hat{s}^i_{i,n-1}. \end{aligned} \tag{2.A14}$$

[10] Recall that (2.21) has shown that the individual trading history $\tilde{x}_{i,1:n-1}$ is redundant information along equilibrium play.

Consequently,

$$\mathbb{E}\left[s_{M,n} - r_{n,n-1}\middle|s_{i,0}, s_{M,1:n-1}, y_{1:n-1}, \tilde{x}_{i,1:n-1}\right] = \psi_n \hat{s}^i_{i,n-1} + \left(\hat{r}^i_{n,n-1} - r_{n,n-1}\right). \tag{2.A15}$$

Similarly,

$$\mathbb{E}\left[v - p_{n-1}\middle|s_{i,0}, s_{M,1:n-1}, y_{1:n-1}, \ldots, \tilde{x}_{i,1:n-1}\right] = \eta_n \hat{s}^i_{i,n-1} + \left(\hat{p}^i_{n-1} - p_{n-1}\right), \tag{2.A16}$$

and

$$\mathbb{E}\left[s_{j,n-1}\middle|s_{i,0}, s_{M,1:n-1}, y_{1:n-1}, \tilde{x}_{i,1:n-1}\right] = \phi_n \hat{s}^i_{i,n-1} + \left(\hat{t}^i_{j,n-1} - t_{j,n-1}\right), \tag{2.A17}$$

where $j \in \{1, \ldots, I\}$ and $j \neq i$. □

2.A.7 Proof of Proposition 2.1

Proof. Given (2.31), informed trader i's Bellman equation (2.5) in trading period $n \in \{1, \ldots, N\}$ can be expressed as

$$\begin{aligned}
&\max_{x_{i,n}} \mathbb{E}\left[(v - p_n)x_{i,n}\middle|s_{i,0}, s_{M,1:n-1}, y_{1:n-1}, \tilde{x}_{i,1:n-1}\right] \\
&+ \mathbb{E}\Big[a_n\left(\hat{s}^i_{i,n}\right)^2 + b_n\left(\hat{t}^i_{j,n} - t_{j,n}\right)^2 + c_n\left(\hat{r}^i_{n+1,n} - r_{n+1,n}\right)^2 + d_n\hat{s}^i_{i,n}\left(\hat{t}^i_{j,n} - t_{j,n}\right) \\
&\quad + e_n\hat{s}^i_{i,n}\left(\hat{r}^i_{n+1,n} - r_{n+1,n}\right) + f_n\left(\hat{t}^i_{j,n} - t_{j,n}\right)\left(\hat{r}^i_{n+1,n} - r_{n+1,n}\right) \\
&\quad + g_n\Big|s_{i,0}, s_{M,1:n-1}, y_{1:n-1}, \tilde{x}_{i,1:n-1}\Big],
\end{aligned} \tag{2.A18}$$

where the market maker's prices and beliefs are given by

$$\begin{aligned}
p_n &= p_{n-1} + \mu_n(s_{M,n} - r_{n,n-1}) + \lambda_n y_n \\
&= p_{n-1} + \mu_n(s_{M,n} - r_{n,n-1}) + \lambda_n\left(x_{i,n} + \sum_{j\neq i}\beta_n s_{j,n-1} + u_n\right),
\end{aligned} \tag{2.A19}$$

and

$$t_{j,n} = t_{j,n-1} + \alpha_n(s_{M,n} - r_{n,n-1}) + \zeta_n y_n$$
$$= t_{j,n-1} + \alpha_n(s_{M,n} - r_{n,n-1}) + \zeta_n \left(x_{i,n} + \sum_{j\neq i} \beta_n s_{j,n-1} + u_n \right), \tag{2.A20}$$

and[11]

$$r_{n+1,n} = r_{n,n-1} + \omega_n(s_{M,n} - r_{n,n-1}) + \delta_n y_n$$
$$= r_{n,n-1} + \omega_n(s_{M,n} - r_{n,n-1}) + \delta_n \left(x_{i,n} + \sum_{j\neq i} \beta_n s_{j,n-1} + u_n \right). \tag{2.A21}$$

By taking the first and second derivatives, one obtains the first-order condition

$$\begin{aligned}
&\mathbb{E}\left[v - p_{n-1} - \mu_n(s_{M,n} - r_{n,n-1}) - \lambda_n \sum_{j\neq i} \beta_n s_{j,n-1} \middle| s_{i,0}, s_{M,1:n-1}, y_{1:n-1}, \tilde{x}_{i,1:n-1} \right] - 2\lambda_n x_{i,n} \\
&+ \mathbb{E}\Big[2b_n \left(\hat{t}^i_{j,n} - t_{j,n} \right)(-\zeta_n) + 2c_n \left(\hat{r}^i_{n+1,n} - r_{n+1,n} \right)(-\delta_n) + d_n \hat{s}^i_{i,n}(-\zeta_n) + e_n \hat{s}^i_{i,n}(-\delta_n) \\
&\quad + f_n \left[-\zeta_n \left(\hat{r}^i_{n+1,n} - r_{n+1,n} \right) - \delta_n \left(\hat{t}^i_{j,n} - t_{j,n} \right) \right] \Big| s_{i,0}, s_{M,1:n-1}, y_{1:n-1}, \tilde{x}_{i,1:n-1} \Big] = 0
\end{aligned} \tag{2.A22}$$

and the second-order condition

$$-2\lambda_n + 2b_n\zeta_n^2 + 2c_n\delta_n^2 + 2f_n\zeta_n\delta_n < 0. \tag{2.A23}$$

Expanding the first-order condition yields

$$\begin{aligned}
&\mathbb{E}\left[v - \hat{p}^i_{n-1} - \mu_n(s_{M,n} - \hat{r}^i_{n,n-1}) - \lambda_n \sum_{j\neq i} \beta_n \hat{s}^i_{j,n-1} \middle| s_{i,0}, s_{M,1:n-1}, y_{1:n-1}, \tilde{x}_{i,1:n-1} \right] \\
&- 2\lambda_n x_{i,n} + \left(\hat{p}^i_{n-1} - p_{n-1} \right) - \mu_n \left(\hat{r}^i_{n,n-1} - r_{n,n-1} \right) + \lambda_n \sum_{j\neq i} \beta_n \left(\hat{s}^i_{j,n-1} - s_{j,n-1} \right) \\
&- \mathbb{E}\Big[(d_n\zeta_n + e_n\delta_n)\hat{s}^i_{i,n} + (2b_n\zeta_n + f_n\delta_n)\left(\hat{t}^i_{j,n} - t_{j,n} \right) \\
&\quad + (2c_n\delta_n + f_n\zeta_n)\left(\hat{r}^i_{n+1,n} - r_{n+1,n} \right) \Big| s_{i,0}, s_{M,1:n-1}, y_{1:n-1}, \tilde{x}_{i,1:n-1} \Big] = 0.
\end{aligned} \tag{2.A24}$$

[11] Since the market maker does not receive a signal after trading period N, it follows that $r_{N+1,N} = 0$.

Recall from (2.29) that

$$\hat{p}^i_{n-1} - p_{n-1} = I\theta_I \left(\hat{t}^i_{j,n-1} - t_{j,n-1}\right) + (N-n+1)\theta_M \left(\hat{r}^i_{n,n-1} - r_{n,n-1}\right). \quad (2.A25)$$

Also, Lemma 2.4 implies

$$\mathbb{E}\left[v - \hat{p}^i_{n-1}\middle| s_{i,0}, s_{M,1:n-1}, y_{1:n-1}, \tilde{x}_{i,1:n-1}\right] = \eta_n \hat{s}^i_{i,n-1}, \quad (2.A26)$$

$$\mathbb{E}\left[\hat{s}^i_{j,n-1}\middle| s_{i,0}, s_{M,1:n-1}, y_{1:n-1}, \tilde{x}_{i,1:n-1}\right] = \phi_n \hat{s}^i_{i,n-1}, \quad (2.A27)$$

$$\mathbb{E}\left[s_{M,n} - \hat{r}^i_{n,n-1}\middle| s_{i,0}, s_{M,1:n-1}, y_{1:n-1}, \tilde{x}_{i,1:n-1}\right] = \psi_n \hat{s}^i_{i,n-1}. \quad (2.A28)$$

Moreover,

$$\begin{aligned} &\mathbb{E}\left[\hat{s}^i_{i,n}\middle| s_{i,0}, s_{M,1:n-1}, y_{1:n-1}, \tilde{x}_{i,1:n-1}\right] \\ =&\hat{s}^i_{i,n-1} - \alpha_n \mathbb{E}\left[s_{M,n} - \hat{r}^i_{n,n-1}\middle|\hat{s}^i_{i,n-1}\right] - \zeta_n \mathbb{E}\left[\beta_n \hat{s}^i_{i,n-1} + \sum_{j\neq i} \beta_n \hat{s}^i_{j,n-1} + u_n \middle| \hat{s}^i_{i,n-1}\right] \\ =&\Big(1 - \alpha_n\psi_n - \zeta_n\beta_n\big[1 + (I-1)\phi_n\big]\Big)\hat{s}^i_{i,n-1}, \end{aligned} \quad (2.A29)$$

and

$$\begin{aligned} &\hat{t}^i_{j,n} - t_{j,n} = \hat{t}^i_{j,n-1} - t_{j,n-1} + \alpha_n\left[s_{M,n} - \hat{r}^i_{n,n-1} - (s_{M,n} - r_{n,n-1})\right] + \zeta_n\left(\hat{y}^i_n - y_n\right) \\ =&\hat{t}^i_{j,n-1} - t_{j,n-1} - \alpha_n\left(\hat{r}^i_{n,n-1} - r_{n,n-1}\right) + \zeta_n\Big[\beta_n\hat{s}^i_{i,n-1} - x_{i,n} - (I-1)\beta_n\left(\hat{t}^i_{j,n-1} - t_{j,n-1}\right)\Big], \end{aligned} \quad (2.A30)$$

and

$$\begin{aligned} &\hat{r}^i_{n+1,n} - r_{n+1,n} = \hat{r}^i_{n,n-1} - r_{n,n-1} + \omega_n\left[s_{M,n} - \hat{r}^i_{n,n-1} - (s_{M,n} - r_{n,n-1})\right] + \delta_n\left(\hat{y}^i_n - y_n\right) \\ =&\hat{r}^i_{n,n-1} - r_{n,n-1} - \omega_n\left(\hat{r}^i_{n,n-1} - r_{n,n-1}\right) + \delta_n\Big[\beta_n\hat{s}^i_{i,n-1} - x_{i,n} - (I-1)\beta_n\left(\hat{t}^i_{j,n-1} - t_{j,n-1}\right)\Big]. \end{aligned} \quad (2.A31)$$

Plugging these relations into the first-order condition and collecting terms yields

$$x_{i,n} = \beta_n\hat{s}^i_{i,n-1} + \gamma_n\left(\hat{t}^i_{j,n-1} - t_{j,n-1}\right) + \rho_n\left(\hat{r}^i_{n,n-1} - r_{n,n-1}\right), \quad (2.A32)$$

where the coefficients β_n, γ_n, and ρ_n are just as in Proposition 2.1. The value function coefficients are derived by plugging the optimal strategy $x_{i,n}$ into the Bellman equation and collecting terms, where the derivation uses the relation

$$\begin{aligned}
&\mathbb{E}\left[\left(\hat{s}^i_{i,n}\right)^2 \middle| s_{i,0}, s_{M,1:n-1}, y_{1:n-1}, \tilde{x}_{i,1:n-1}\right] \\
=&\Big(1-\alpha_n\psi_n-\zeta_n\beta_n\big[1+(I-1)\phi_n\big]\Big)^2\left(\hat{s}^i_{i,n-1}\right)^2 \\
&+\zeta_n^2\Sigma^u_0+\alpha_n^2 \operatorname{Var}\left[s_{M,n}-\hat{r}^i_{n,n-1}\middle| s_{i,0}, s_{M,1:n-1}, \hat{y}^i_{1:n-1}\right] \\
&+2(I-1)\zeta_n\beta_n\alpha_n \operatorname{Cov}\left[\hat{s}^i_{j,n-1}, s_{M,n}-\hat{r}^i_{n,n-1}\middle| s_{i,0}, s_{M,1:n-1}, \hat{y}^i_{1:n-1}\right] \\
&+(I-1)\zeta_n^2\beta_n^2\Big(\operatorname{Var}\left[\hat{s}^i_{j,n-1}\middle| s_{i,0}, s_{M,1:n-1}, \hat{y}^i_{1:n-1}\right] \\
&+(I-2)\operatorname{Cov}\left[\hat{s}^i_{j,n-1}, \hat{s}^i_{k,n-1}\middle| s_{i,0}, s_{M,1:n-1}, \hat{y}^i_{1:n-1}\right]\Big).
\end{aligned} \tag{2.A33}$$

Since the remaining proof assumes equilibrium play, the "hat" notation is omitted. The regression coefficients ϕ_n and ψ_n are defined by the projection theorem:

$$\phi_n = \frac{\operatorname{Cov}\left[s_{j,n-1}, s_{i,n-1}\right]}{\operatorname{Var}\left[s_{i,n-1}\right]} = \frac{\Sigma^{s_{j,0},s_{i,0}}_{n-1}}{\Sigma^{s_{i,0}}_{n-1}}, \tag{2.A34}$$

$$\psi_n = \frac{\operatorname{Cov}\left[s_{M,n}-r_{n,n-1}, s_{i,n-1}\right]}{\operatorname{Var}\left[s_{i,n-1}\right]} = \frac{\Sigma^{s_{M,n},s_{i,0}}_{n-1}}{\Sigma^{s_{i,0}}_{n-1}}, \tag{2.A35}$$

while the regression coefficient η_n is computed as follows:

$$\begin{aligned}
\eta_n &= \frac{\operatorname{Cov}\left[v-p_{n-1}, s_{i,n-1}\right]}{\operatorname{Var}\left[s_{i,n-1}\right]} \\
&= \frac{\operatorname{Cov}\left[\hat{v}-p_{n-1}, s_{i,n-1}\right]}{\operatorname{Var}\left[s_{i,n-1}\right]} \\
&= \frac{\operatorname{Cov}\left[\hat{v}-\theta_I\sum_{j=1}^I t_{j,n-1}-\theta_M\left[\sum_{t=1}^{n-1} s_{M,t}+\sum_{t=n}^N r_{t,n-1}\right], s_{i,n-1}\right]}{\operatorname{Var}\left[s_{i,n-1}\right]} \\
&= \frac{\operatorname{Cov}\left[\theta_I\sum_{j=1}^I s_{j,n-1}+\theta_M\sum_{t=n}^N (s_{M,t}-r_{t,n-1}), s_{i,n-1}\right]}{\operatorname{Var}\left[s_{i,n-1}\right]} \\
&= \frac{\theta_I\left[\Sigma^{s_{i,0}}_{n-1}+(I-1)\Sigma^{s_{j,0},s_{i,0}}_{n-1}\right]+(N-n+1)\theta_M\Sigma^{s_{M,n},s_{i,0}}_{n-1}}{\Sigma^{s_{i,0}}_{n-1}} \\
&= \theta_I\big[1+(I-1)\phi_n\big]+(N-n+1)\theta_M\psi_n.
\end{aligned} \tag{2.A36}$$

Moreover, note that

$$
\begin{aligned}
&\operatorname{Var}\left[s_{M,n} - r_{n,n-1} \middle| s_{i,0}, s_{M,1:n-1}, y_{1:n-1}\right] \\
=&\,\mathbb{E}\left[\left(s_{M,n} - r_{n,n-1} - \psi_n s_{i,n-1}\right)^2\right] \\
=&\,\mathbb{E}\left[\left(s_{M,n} - r_{n,n-1}\right)^2\right] - 2\psi_n\,\mathbb{E}\left[\left(s_{M,n} - r_{n,n-1}\right)s_{i,n-1}\right] + \psi_n^2\,\mathbb{E}\left[\left(s_{i,n-1}\right)^2\right] \\
=&\,\Sigma_{n-1}^{s_{M,n}} - 2\psi_n\Sigma_{n-1}^{s_{M,n},s_{i,0}} + \psi_n^2\Sigma_{n-1}^{s_{i,0}} \\
=&\,\Sigma_{n-1}^{s_{M,n}} - \psi_n^2\Sigma_{n-1}^{s_{i,0}},
\end{aligned}
\tag{2.A37}
$$

and

$$
\begin{aligned}
&\operatorname{Var}\left[s_{j,n-1} \middle| s_{i,0}, s_{M,1:n-1}, y_{1:n-1}\right] \\
=&\,\mathbb{E}\left[\left(s_{j,n-1} - \phi_n s_{i,n-1}\right)^2\right] \\
=&\,\mathbb{E}\left[\left(s_{j,n-1}\right)^2\right] - 2\phi_n\,\mathbb{E}\left[s_{j,n-1}s_{i,n-1}\right] + \phi_n^2\,\mathbb{E}\left[\left(s_{i,n-1}\right)^2\right] \\
=&\,\Sigma_{n-1}^{s_{j,0}} - 2\phi_n\Sigma_{n-1}^{s_{j,0},s_{i,0}} + \phi_n^2\Sigma_{n-1}^{s_{i,0}} \\
=&\,(1-\phi_n^2)\Sigma_{n-1}^{s_{i,0}},
\end{aligned}
\tag{2.A38}
$$

and

$$
\begin{aligned}
&\operatorname{Cov}\left[s_{j,n-1}, s_{k,n-1} \middle| s_{i,0}, s_{M,1:n-1}, y_{1:n-1}\right] \\
=&\,\operatorname{Cov}\left[s_{j,n-1} - \phi_n s_{i,n-1}, s_{k,n-1} - \phi_n s_{i,n-1}\right] \\
=&\,\operatorname{Cov}\left[s_{j,n-1}, s_{i,n-1}\right] - 2\phi_n\operatorname{Cov}\left[s_{j,n-1}, s_{i,n-1}\right] + \phi_n^2\operatorname{Cov}\left[s_{i,n-1}, s_{i,n-1}\right] \\
=&\,\Sigma_{n-1}^{s_{j,0},s_{i,0}} - 2\phi_n\Sigma_{n-1}^{s_{j,0},s_{i,0}} + \phi_n^2\Sigma_{n-1}^{s_{i,0}} \\
=&\,(1-\phi_n)\Sigma_{n-1}^{s_{j,0},s_{i,0}},
\end{aligned}
\tag{2.A39}
$$

and

$$\begin{aligned}
&\operatorname{Cov}\left[s_{j,n-1}, s_{M,n} - r_{n,n-1} \middle| s_{i,0}, s_{M,1:n-1}, y_{1:n-1}\right] \\
=&\operatorname{Cov}\left[s_{j,n-1} - \phi_n s_{i,n-1}, s_{M,n} - r_{n,n-1} - \psi_n s_{i,n-1}\right] \\
=&\operatorname{Cov}\left[s_{j,n-1}, s_{M,n} - r_{n,n-1}\right] - \phi_n \operatorname{Cov}\left[s_{i,n-1}, s_{M,n} - r_{n,n-1}\right] \\
&- \psi_n \operatorname{Cov}\left[s_{j,n-1}, s_{i,n-1}\right] + \phi_n\psi_n \operatorname{Cov}\left[s_{i,n-1}, s_{i,n-1}\right] \\
=&\Sigma_{n-1}^{s_{i,0},s_{M,n}} - \phi_n \Sigma_{n-1}^{s_{i,0},s_{M,n}} - \psi_n \Sigma_{n-1}^{s_{j,0},s_{i,0}} + \phi_n\psi_n \Sigma_{n-1}^{s_{i,0}} \\
=&(1-\phi_n)\Sigma_{n-1}^{s_{i,0},s_{M,n}}.
\end{aligned} \tag{2.A40}$$

To compute the projection coefficients α_n, ζ_n, ω_n, and δ_n, recall that

$$y_n = \sum_{i=1}^{I} \beta_n s_{i,n-1} + u_n, \tag{2.A41}$$

and define for all $n \in \{1, \ldots, N-1\}$:

$$\Sigma_{n-1}^{y_n} \equiv \operatorname{Var}\left[y_n \middle| s_{M,1:n-1}, y_{1:n-1}\right] = I\beta_n^2 \left[\Sigma_{n-1}^{s_{i,0}} + (I-1)\Sigma_{n-1}^{s_{i,0},s_{j,0}}\right] + \Sigma_0^u, \tag{2.A42}$$

$$\Sigma_{n-1}^{s_{i,0},y_n} \equiv \operatorname{Cov}\left[s_{i,0}, y_n \middle| s_{M,1:n-1}, y_{1:n-1}\right] = \beta_n \left[\Sigma_{n-1}^{s_{i,0}} + (I-1)\Sigma_{n-1}^{s_{i,0},s_{j,0}}\right], \tag{2.A43}$$

$$\Sigma_{n-1}^{s_{M,n},y_n} \equiv \operatorname{Cov}\left[s_{M,n}, y_n \middle| s_{M,1:n-1}, y_{1:n-1}\right] = I\beta_n \Sigma_{n-1}^{s_{i,0},s_{M,n}}, \tag{2.A44}$$

$$\Sigma_{n-1}^{s_{M,n+1},y_n} \equiv \operatorname{Cov}\left[s_{M,n+1}, y_n \middle| s_{M,1:n-1}, y_{1:n-1}\right] = I\beta_n \Sigma_{n-1}^{s_{i,0},s_{M,n}}. \tag{2.A45}$$

Then the projection theorem implies[12]

$$\begin{pmatrix} \alpha_n \\ \zeta_n \end{pmatrix} = \begin{pmatrix} \Sigma_{n-1}^{s_{M,n}} & \Sigma_{n-1}^{s_{M,n},y_n} \\ \Sigma_{n-1}^{s_{M,n},y_n} & \Sigma_{n-1}^{y_n} \end{pmatrix}^{-1} \begin{pmatrix} \Sigma_{n-1}^{s_{i,0},s_{M,n}} \\ \Sigma_{n-1}^{s_{i,0},y_n} \end{pmatrix}, \tag{2.A46}$$

$$\begin{pmatrix} \omega_n \\ \delta_n \end{pmatrix} = \begin{pmatrix} \Sigma_{n-1}^{s_{M,n}} & \Sigma_{n-1}^{s_{M,n},y_n} \\ \Sigma_{n-1}^{s_{M,n},y_n} & \Sigma_{n-1}^{y_n} \end{pmatrix}^{-1} \begin{pmatrix} \Sigma_{n-1}^{s_{M,n+1},s_{M,n}} \\ \Sigma_{n-1}^{s_{M,n+1},y_n} \end{pmatrix}. \tag{2.A47}$$

[12] Since the market maker does not receive a signal after trading period N, it follows that $\omega_N = 0$ and $\delta_N = 0$.

Finally, for all $n \in \{1, \dots, N-1\}$, define the conditional covariance matrix $\boldsymbol{\Sigma_n}$ as

$$\boldsymbol{\Sigma_n} \equiv \begin{pmatrix} \Sigma_n^{s_{1,0}} & \dots & \Sigma_n^{s_{1,0},s_{I,0}} & \Sigma_n^{s_{1,0},s_{M,n+1}} & \dots & \Sigma_n^{s_{1,0},s_{M,N}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \Sigma_n^{s_{I,0},s_{1,0}} & \dots & \Sigma_n^{s_{I,0}} & \Sigma_n^{s_{I,0},s_{M,n+1}} & \dots & \Sigma_n^{s_{I,0},s_{M,N}} \\ \Sigma_n^{s_{M,n+1},s_{1,0}} & \dots & \Sigma_n^{s_{M,n+1},s_{I,0}} & \Sigma_n^{s_{M,n+1}} & \dots & \Sigma_n^{s_{M,n+1},s_{M,N}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \Sigma_n^{s_{M,N},s_{1,0}} & \dots & \Sigma_n^{s_{M,N},s_{I,0}} & \Sigma_n^{s_{M,N},s_{M,n+1}} & \dots & \Sigma_n^{s_{M,N}} \end{pmatrix}, \tag{2.A48}$$

where the entries are conditional on the public histories $(s_{M,1:n}, y_{1:n})$. Let $\boldsymbol{\Sigma_{n-1}}$ be the corresponding covariance matrix where the entries are conditional on the public histories $(s_{M,1:n-1}, y_{1:n-1})$. Since long-lived and future short-lived signals are jointly normally distributed, it follows that

$$\boldsymbol{\Sigma_n} = \boldsymbol{\Sigma_{n-1}} - \begin{pmatrix} \Sigma_{n-1}^{s_{1,0},s_{M,n}} & \Sigma_{n-1}^{s_{1,0},y_n} \\ \vdots & \vdots \\ \Sigma_{n-1}^{s_{I,0},s_{M,n}} & \Sigma_{n-1}^{s_{I,0},y_n} \\ \Sigma_{n-1}^{s_{M,n+1},s_{M,n}} & \Sigma_{n-1}^{s_{M,n+1},y_n} \\ \vdots & \vdots \\ \Sigma_{n-1}^{s_{M,N},s_{M,n}} & \Sigma_{n-1}^{s_{M,N},y_n} \end{pmatrix} \begin{pmatrix} \Sigma_{n-1}^{s_{M,n}} & \Sigma_{n-1}^{s_{M,n},y_n} \\ \Sigma_{n-1}^{s_{M,n},y_n} & \Sigma_n^{y_n} \end{pmatrix}^{-1} \begin{pmatrix} \Sigma_{n-1}^{s_{1,0},s_{M,n}} & \Sigma_{n-1}^{s_{1,0},y_n} \\ \vdots & \vdots \\ \Sigma_{n-1}^{s_{I,0},s_{M,n}} & \Sigma_{n-1}^{s_{I,0},y_n} \\ \Sigma_{n-1}^{s_{M,n+1},s_{M,n}} & \Sigma_{n-1}^{s_{M,n+1},y_n} \\ \vdots & \vdots \\ \Sigma_{n-1}^{s_{M,N},s_{M,n}} & \Sigma_{n-1}^{s_{M,N},y_n} \end{pmatrix}', \tag{2.A49}$$

yielding the desired block structure for conditional (co)variances. □

2.A.8 Algorithm

The algorithm for computing the equilibrium in Proposition 2.1 is similar to Gounas (2021). Input parameters are the number of informed traders I, the number of trading periods N, the period zero (co)variances $\Sigma_0^{\hat{v}}$, $\Sigma_0^{v,s_{i,0}}$, $\Sigma_0^{v,s_{M,n}}$, $\Sigma_0^{s_{i,0}}$, $\Sigma_0^{s_{M,n}}$, $\Sigma_0^{s_{i,0},s_{M,n}}$, Σ_0^{u}, and the constants χ_I and χ_M defined in (2.17) and (2.18), respectively. For the input parameters to be well-defined, the covariance matrix in (2.1) must be positive definite. If this condition is satisfied, one can compute θ_I and θ_M as in Appendix 2.A.1.

Using backward induction, provide initial guesses for the period $N-1$ conditional (co)variances $\Sigma_{N-1}^{s_{i,0}}$, $\Sigma_{N-1}^{s_{M,N}}$, and $\Sigma_{N-1}^{s_{i,0},s_{M,n}}$.[13] Given χ_I, one can immediately compute $\Sigma_{N-1}^{s_{i,0},s_{j,0}}$ from (2.17). Since the value function coefficients $a_N, \ldots, g_N$ and the projection coefficients ω_N and δ_N are zero in period N, it follows that

$$\beta_N = \frac{I\beta_N^2\left[\Sigma_{N-1}^{s_{i,0}} + (I-1)\Sigma_{N-1}^{s_{i,0},s_{j,0}}\right] + \Sigma_0^u}{I\beta_N\left[2\Sigma_{N-1}^{s_{i,0}} + (I-1)\Sigma_{N-1}^{s_{i,0},s_{j,0}}\right]}. \tag{2.A50}$$

Solving this quadratic equation for β_N yields the solutions $-\sqrt{\Sigma_0^u/(I\Sigma_{N-1}^{s_{i,0}})}$ and $\sqrt{\Sigma_0^u/(I\Sigma_{N-1}^{s_{i,0}})}$, where only the positive root satisfies the second-order condition. Given β_N, one can compute α_N, ζ_N, μ_N, λ_N, γ_N, ρ_N, and the value function coefficients $a_{N-1}, \ldots, g_{N-1}$.

Iterating backward over every period $n \in \{1, \ldots, N-1\}$, input parameters are β_{n+1}, $\Sigma_n^{s_{i,0}}$, $\Sigma_n^{s_{M,n+1}}$, and the value function coefficients $a_n, \ldots, g_n$. With these inputs, one can simultaneously solve for β_n, $\Sigma_{n-1}^{s_{i,0}}$, $\Sigma_{n-1}^{s_{M,n}}$, and $\Sigma_{n-1}^{s_{i,0},s_{M,n}}$. Immediately, $\Sigma_{n-1}^{s_{i,0},s_{j,0}}$ and $\Sigma_{n-1}^{s_{M,n},s_{M,n+1}}$ follow from (2.17) and (2.18), respectively. Then it is straightforward to compute α_n, ζ_n, ω_n, δ_n, μ_n, λ_n, γ_n, ρ_n, and the value function coefficients $a_{n-1}, \ldots, g_{n-1}$, allowing one to proceed to the next iteration.

The algorithm terminates if the period zero (co)variances satisfy the initial specification, the second-order condition holds in each period, and conditional (co)variances and the value function are consistent over time. Otherwise, the algorithm runs anew with adjusted initial guesses for $\Sigma_{N-1}^{s_{i,0}}$, $\Sigma_{N-1}^{s_{M,N}}$, and $\Sigma_{N-1}^{s_{i,0},s_{M,n}}$.

[13] The period N conditional (co)variances are not necessary to compute the equilibrium in Proposition 2.1.

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